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INTEGRAL EQUATION METHODS FOR THE NUMERICAL SOLUTION OF FREE SURFACE PROBLEMS IN INVISCID FLOW.

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INTEGRAL EQUATION METHODS FOR THE NUMERICAL SOLUTION OF FREE SURFACE PROBLEMS IN INVISCID FLOW

by

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ABSTRACT

This dissertation is a study of integral equation methods for the numerical solution of free surface problems in inviscid flow. The flow is assumed to be two-dimensional, steady and incompressible and the motion irrotational. All problems studied here include the influence of gravity. Two methods, the Cauchy integral equation method and the Riemann-Hilbert method for a mixed-boundary-value problem, will be introduced. Four problems, the steady water wave problem, a two-dimensional vertical jet from a rectangular vessel, a two dimensional vertical jet from an infinite channel and the solitary water wave problem, will be solved. The first two problems are solved by the Cauchy integral method and the other two by the Riemann-Hilbert method. Each problem has been programmed and run on a computer, and the computed results plotted and compared with those of the authors quoted in Chapter I, whenever possible.

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NOMENCLATURE

p = pressure

p_E = pressure at the point E

p_1 = pressure at infinity

q = magnitude of velocity

q_E = magnitude of velocity at the point E

q_1 = magnitude of velocity at infinity

g = gravity

y = vertical coordinate in the physical plane

W = complex velocity potential

ϕ = velocity potential

ψ = stream function

u = horizontal coordinate of velocity

v = vertical coordinate of velocity

$U+iV$ = an analytic function

Γ = a closed contour

F^2 = Froude number

C_p = excess pressure coefficient

ρ = constant density of fluid

t = auxiliary half plane

t_i = real t-plane coordinate

ζ = normalized conjugate complex velocity

ω = log ζ

θ = argument of complex velocity

Q(t) = solution of Riemann-Hilbert problem

H(t) = solution of the homogeneous Riemann-Hilbert problem

c = wave speed

λ = wavelength

a = amplitude of wave

T_1 = time required for a fluid particle to travel half-wavelength along the free surface

u_0 = mass transport velocity

<u>NOTE</u>: We use the numbering system as follows: for example , TABLE 5.2 refers to Chapter 5, and FIGURE 4.2 to Chapter 4.

CHAPTER I

INTRODUCTION

The problem of free streamlines is an old and difficult one in hydrodynamics. The difficulty arises from the fact that it is a mixed Dirichlet-Neumann-boundary-value problem in which the position(s) of part of the boundary, the free surface(s), is (are) unknown, and along the free streamline, the pressure is constant. For general theory of the subject, the reader should consult Milne-Thomson [35], and Birkhoff and Zarantonello [3]. For more complete bibliography on this subject, see Cryer [11].

STEADY WATER WAVE PROBLEM

The problem of a steady progressive wave is such a free surface problem. The theory of water waves was first investigated by Gerstner [14] in 1804. In 1846 Stokes [49] investigated the motion of a water wave of constant form and finite amplitude to third order of approximation, for water of infinite depth, and to second order of approximation for finite depth. Later, in 1880 [50] he reworked the problem, using the velocity potential and stream function, rather than the space coordinates, as independent variables. He obtained the fifth order of approximation for infinite depth and the third order of approximation for finite depth. De [12] extended the works of Stokes to fifth order of approximation. Recently (1974), Schwartz [42] extended the approximation to very high order (30 to 115 terms) using a modern digital computer to perform the coefficient arithmetic, and used Padé approximation to sum the series and continue analytically.

A proof of the existence of periodic gravity waves of constant form in water of infinite depth was first given by Nekrasov [37] and later independently by Levi-Civita [30]. Nekrasov formulated the problem as a non-linear integral equation, and showed that a non-trivial solution could be found for sufficiently small values of wave amplitudes. Levi-Civita's formulation of the problem is essentially the same as Nekrasov's. He established the existence of the solution by establishing the convergence of a series in amplitude-to-wavelength μ, for sufficiently small values of μ. No estimate of a radius of convergence was given. Struik [51] extended the work of Levi-Civita to water of finite depth. Some errors of Struik's paper have been corrected by Hunt [19]. Krasovskii [23] in 1960 gave a proof using theory of positive operators. He showed that Nekrasov's integral equation has a solution for any depth provided that the maximum surface angle does not exceed $\frac{\pi}{6}$.

The existence of highest progressive wave is still an open problem. In 1880, Stokes [49] assuming that such a wave existed, showed that the maximum surface angle should be $\frac{\pi}{6}$. Michell [33], Havelock [17], Nekrasov [37] and Yamada [57] also investigated the highest progressive waves, and Michell [33] showed using series, that the amplitude-to-wavelength ratio is approximately 1:7.

For more detail of discussions of the subject, see Wehausen and Laitone [55], Lamb [24], Milne-Thomson [35] and Stoker [48].

SOLITARY WAVE PROBLEM

The problem of a solitary wave was first investigated experimentally by Russell [41] in 1844. From his experimental studies, he found that waves travelling into a region of decreasing depth gradually increase in sharpness of crest, finally breaking when the wave amplitude a was approximately equal to the depth of undisturbed fluid v_1. In 1871, Boussinesq [4] investigated the problem neglecting the effects of slope, resistance, and velocity distribution, and later independently Rayleigh [39] did likewise. Boussinesq obtained the basic equations of acceleration and continuity for the given boundary conditions by making one simplifying assumption after another. Rayleigh's approach was mathematical, involving the integration of the function $W = \phi + i\psi$. From Rayleigh's equation, it was found that the maximum possible amplitude of a solitary wave could be reached when $a = v_1$. In 1926, Weinstein [56] investigated the solitary wave using the method of Levi-Civita [30]. He obtained the equation for wave speed $c = q_1$ only slightly different from the Rayleigh-Boussinesq expression.

In 1966, Schwitters [43] obtained the solution for the solitary water wave as a non-linear integral equation, similar to that of Nekrasov's [37]. In 1976, Spielvogel and Spielvogel [47] also investigated the solitary wave. They first constructed the solutions using a complex difference-differential equation and the Lewy method [31] for reflection across the free surface, and then showed the convergence.

For more detail of discussion of this subject, see Milne-Thomson [35], Lamb [24], Rouse [40] and Stoker [48].

JET THEORY

The first problem of jet theory was investigated and solved by Helmholtz [18] in 1868. In 1869, Kirchhoff [22] generalized Helmholtz's method. They considered a jet flow issuing from a slot in an unbounded plane, with gravity neglected. Helmholtz and Kirchhoff mapped the whole fluid region onto an upper half-disk with free surfaces corresponding to the semi-circle. To find the solution, they introduced and applied the theory of functions of a complex variable. In 1890, Joukowski [21] modified Kirchhoff's method by introducing an intermediate parameter, defined on the upper half-plane, in the case when the fluid region is simply-connected, and the solid walls forming the boundary of the fluid region consist of a finite number of straight lines. In 1907 Levi-Civita [29] generalized Kirchhoff's method to the flow around two contours. To some extent, Joukowski 's method is more general than Levi-Civita's method. The first specific problem of the flow with wake around a curved obstacle—a circular arc was considered and solved by Nekrasov [38] in 1922. He obtained the solution in terms of a non-linear integral equation, and applied the method of successive approximations to prove the convergence and uniqueness of the solution for small values of a parameter proportional to the central angle subtended by the arc. Conway [9] in 1967 considered an infinite channel of infinite height bounded by two horizontal planes. When a transverse aperture is made on the lower plane, the fluid will emerge as a jet bounded by two free surfaces. He expressed the solution as a non-linear integral equation, similar to that of Nekrasov, and solved the integral equation using an iterative method.

For more detail of discussions of the subject, see Milne-Thomson [35], Birkhoff and Zarantonello [3], Gurevich [16] and Sedov [44].

OUTLINE OF PRESENT WORK

In Chapter 2, we introduce two methods, the Cauchy integral equation method and Riemann-Hilbert method for a mixed-boundary-value problem, for solving some free surface problems. In Chapter 3, we present the numerical methods for solving the free surface problems using these two methods. In Chapter 4, we study the steady water wave problem using Cauchy integral equation method. In Chapter 5, we solve the two dimensional vertical jet from a rectangular vessel using the Cauchy integral equation method. In Chapters 6 and 7, we investigate the two-dimensional vertical jet from an infinite channel and the solitary water wave respectively, using the Riemann-Hilbert method.

All problems considered here are under gravity. Each problem has been programmed and run on a computer, and the computed results plotted and compared with those of the authors quoted above, whenever possible.

CHAPTER II

GENERAL THEORY

2.1 Cauchy Integral Equation Method

We consider steady, two-dimensional, irrotational flow of an inviscid, incompressible fluid; for example, the steady water wave problem in a channel. Suppose the fluid region under consideration is finite and bounded, either by means of the symmetry of the flow or cut off at a finite distance upstream and downstream. Part of the boundary is a free surface, and the effect of gravity is taken into account.

Bernoulli's equation along a streamline can be written as,

$$\frac{p}{\rho} + \frac{1}{2} q^2 + gy = \text{constant} , \tag{2.1}$$

where p is the pressure, ρ is the fluid density, q is the speed of the fluid, y is the vertical distance between a point and some reference elevation, and g is the acceleration due to gravity.

Let $z = x + iy$ and $W = \phi + i \psi$ where ϕ is the velocity potential and ψ the stream function. Then $\frac{dW}{dz} = u - iv = q e^{-i\theta}$. We then map the fluid region in the physical plane onto a rectangle in the complex potential plane, the W-plane, such that the two horizontal sides correspond to $\psi = 0$ and $\psi = \psi_1 =$

constant, and the vertical sides to $\phi = 0$ and $\phi = \phi_1$ = constant. Bernoulli's equation (2.1) along the free surface, which lies either on $\psi = 0$ or $\psi = \psi_1$, with density $\rho = 1$, will lead to

$$q^2 + 2\,gy = q_0^2 = \text{constant}\,, \tag{2.2}$$

where q_o is the speed at a point on the free surface at which $y = 0$. This may be written in integral form by differentiating with respect to ϕ, replacing $\frac{dy}{d\phi}$ by $\frac{v}{q^2} = \frac{\sin\theta}{q}$ and integrating to obtain

$$q^3 + 3g\int_0^{\phi} \sin\theta(t)\ dt = q_0^3 \tag{2.3}$$

We introduce dimensionless variables*

$$\overline{W} = \frac{W}{\psi_1}\ ,\ \ \overline{q} = \frac{q}{q_0}\ ,\ \ \overline{g} = \frac{g\psi_1}{q_0^{\ 3}} \tag{2.4}$$

so that (2.3) becomes

$$\overline{q}^3 + 3\overline{g}\int_0^{\overline{\phi}} \sin\theta(t)\ dt = \overline{q}_0^3 = 1$$

Now, dropping all the bars, we write

$$q^3 + 3g\int_0^{\phi} \sin\theta(t)\ dt = q_0^3 = 1\,, \tag{2.5}$$

remembering that <u>all variables in the water wave problem</u>

---

*Since we can write $\overline{g} = \frac{gd}{q_0^2}$, where $\psi_1 = q_0 d$, $1/\overline{g}$ corresponds to a Froude number.

hereafter are dimensionless. Then $q_0 = 1$ and $\psi_1 = 1$. Equation (2.5) is the free surface condition in integral form.

If all q's along all sides of the rectangle and θ along the free surface have been computed, then the shape of the free surface (x,y) and other quantities may be computed. Clearly, from (2.2), all y's are known. Then x's and y's can also be computed from $x_\phi = \frac{\cos\theta}{q}$ and $y_\phi = \frac{\sin\theta}{q}$, that is,

$$\left\{ \begin{aligned} x(\phi) &= \int_0^\phi \frac{\cos\theta(t)\,dt}{q(t)} , \\ y(\phi) &= \int_0^\phi \frac{\sin\theta(t)\,dt}{q(t)} , \end{aligned} \right. \quad \text{along the free surface} \qquad (2.6)$$

CAUCHY INTEGRAL EQUATIONS

Next, we investigate the Cauchy integral formula. Let Γ be a simple closed contour, taken in the positive sense (counterclockwise) such that the function f(z) is analytic at every point on and inside Γ. Then the Cauchy integral formula is

$$2\pi i\ f(z_0) = \int_\Gamma \frac{f(z)\,dz}{z-z_0} , \qquad (2.7)$$

if z_0 is an interior point.

The left hand side of (2.7) is zero if z_0 is an exterior point. Now, suppose z_0 is a point on the contour Γ, and Γ is smooth at z_0, then the Cauchy integral formula becomes

$$\pi i f(z_0) = \int_\Gamma \frac{f(z)dz}{z-z_0} \tag{2.8}$$

However, when the slope of Γ is discontinuous at z_0, with interior angle β, the Cauchy integral formula assumes the form

$$\beta i\ f(z_0) = \int_\Gamma \frac{f(z)dz}{z-z_0}$$

In particular, when $\beta = \frac{\pi}{2}$, we obtain the formula at a corner of a rectangle

$$\frac{\pi}{2} i\ f(z_0) = \int_\Gamma \frac{f(z)dz}{z-z_0} \tag{2.9}$$

Let $f(z) = U(x,y) + i\ V(x,y)$ and $z - z_0 = \rho e^{i\alpha}$. Let s be the arc length along the contour Γ and $\underset{\sim}{n}$ the inward unit normal. Then the Cauchy-Riemann conditions are

$$\begin{cases} \dfrac{\partial U}{\partial s} = \dfrac{\partial V}{\partial n}\ , \quad \dfrac{\partial U}{\partial n} = -\dfrac{\partial V}{\partial s}\ , \\[2ex] \dfrac{\partial (\ln \rho)}{\partial s} = \dfrac{\partial \alpha}{\partial n}\ , \quad \dfrac{\partial (\ln \rho)}{\partial n} = -\dfrac{\partial \alpha}{\partial s} \end{cases} \tag{2.10}$$

Now, $z - z_0 = \rho e^{i\alpha}$, hence $dz = (z - z_0)\ [d(\ln \rho) + i d\alpha]$ and equation (2.8) becomes

$$\pi\ [iU(x_0,y_0) - V(x_0,y_0)] = \int_\Gamma [U(x,y) + iV(x,y)][d(\ln \rho) + i d\alpha]$$

Equating the real and imaginary parts, we obtain

$$\begin{cases} \pi U(x_0,y_0) = \displaystyle\int_\Gamma V(x,y)d(\ln \rho) + \int_\Gamma U(x,y)d\alpha\ , \\[2ex] \pi V(x_0,y_0) = -\displaystyle\int_\Gamma U(x,y)d(\ln \rho) + \int_\Gamma V(x,y)d\alpha . \end{cases} \tag{2.11}$$

These integral equations were used by Lauck [28] to obtain the flow over a weir of infinite depth and extent, using graphical integration. Integrating the first integrals of both equations of (2.11) by parts, and using (2.10), we obtain

$$\left\{\begin{aligned} \pi U(x_0,y_0) &= \int_\Gamma \frac{\partial U(x,y)}{\partial n}\,\ell n\rho\,ds - \int_\Gamma U(x,y)\,\frac{\partial(\ell n\rho)}{\partial n}\,ds,\\ \pi V(x_0,y_0) &= \int_\Gamma \frac{\partial V(x,y)}{\partial n}\,\ell n\rho\,ds - \int_\Gamma V(x,y)\,\frac{\partial(\ell n\rho)}{\partial n}\,ds. \end{aligned}\right. \tag{2.12}$$

Equations (2.12) are of the same form as the equation of Jaswon [20] (equation (18) of [20]) and also Green's boundary formula. Applying (2.10) again, we find that equations (2.11) and (2.12) can be rewritten in the following form

$$\left\{\begin{aligned} \pi U(x_0,y_0) &= -\int_\Gamma \frac{\partial V(x,y)}{\partial s}\,\ell n\rho\,ds + \int_\Gamma U(x,y)\frac{\partial\alpha}{\partial s}\,ds,\\ \pi V(x_0,y_0) &= \int_\Gamma \frac{\partial U(x,y)}{\partial s}\,\ell n\rho\,ds + \int_\Gamma V(x,y)\frac{\partial\alpha}{\partial s}\,ds. \end{aligned}\right. \tag{2.13}$$

For a corner point z_0, (2.9) will become

$$\left\{\begin{aligned} \frac{\pi}{2}\,U(x_0,y_0) &= -\int_\Gamma \frac{\partial V(x,y)}{\partial s}\,\ell n\rho\,ds + \int_\Gamma U(x,y)\frac{\partial\alpha}{\partial s}\,ds,\\ \frac{\pi}{2}\,V(x_0,y_0) &= \int_\Gamma \frac{\partial U(x,y)}{\partial s}\,\ell n\rho\,ds + \int_\Gamma V(x,y)\frac{\partial\alpha}{\partial s}\,ds. \end{aligned}\right. \tag{2.14}$$

SOLUTION AND DISCUSSIONS

In applying the above to our problems, we replace, in both equations (2.13) and (2.14), x by ϕ, y by ψ, ρ^2 by $(\phi-\phi_0)^2$

$+ (\psi - \psi_0)^2$, $\tan\alpha$ by $\frac{\psi - \psi_0}{\phi - \phi_0}$, s by ϕ or ψ, and Γ by the rectangle in the w-plane. Note that $W = \phi + i\psi$ and $W_0 = \phi_0 + i\psi_0$ are both points on the rectangle Γ. Making use of the boundary conditions (for example, θ = constant along some parts of sides of the rectangle), equations (2.13) and (2.14) together with free surface condition (2.5) will solve our problems. However it is not possible to solve the problem analytically, since (2.13), (2.14) and (2.5) involve some unknown functions. Hence numerical methods have to be employed. Suppose that we choose N and M points on each horizontal and vertical sides of the rectangle respectively, and the integrals in (2.13) and (2.14) are approximated by some quadrature formulas. Then substituting the values of U, V, $\frac{\partial U}{\partial s}$ and $\frac{\partial V}{\partial s}$ at these points into equations (2.13) and (2,14), we have $2(M + N - 1)$ algebraic linear equations. Moreover, suppose that from the boundary conditions and free surface condition (2.5), half of these $U(x_0, y_0)$ and $V(x_0, y_0)$ (and hence $\frac{\partial U(x_0, y_0)}{\partial s}$ and $\frac{\partial V(x_0, y_0)}{\partial s}$) are known; then we still have $(M + N - 1)$ linear equations in $(M + N - 1)$ unknowns. Direct methods for solving this linear system are not practical particularly when N and M are large. Iterative methods, particularly the Gauss-Seidel method, are found to be most convenient.

It is natural to ask how the Cauchy integral equation method compares with other numerical methods, for example, the finite difference method and finite element method. Suppose we take, for example, 16 points on each horizontal side, and 11 points on vertical side of the rectangle, and divide the rectangle into 150 small rectangles. Then there are 150 elements (in fluid region) in the z-plane. Consider the water wave problem. Using boundary conditions and free surface condition, for both finite difference and finite element methods, we have to solve approximately 300 equations, but only 48 equations using the Cauchy method.

In the next chapter, we develop a numerical method for solving the singular integral equations, particularly some approximate formulas for singularities and points near a corner. In the later chapters, we apply the Cauchy integral equation method to two free surface problems: the steady water wave problem and the vertical jet problem, both under gravity. In the former, we use $f(z) = U(x,y) + iV(x,y) = q + i(-\theta)$, and latter $f(z) = U(x,y) + iV(x,y) = u + i(-v)$.

2.2 Riemann-Hilbert Problem

Consider steady, two dimensional, irrotational flow of an inviscid, incompressible fluid. For example, a channel jet with an aperture on the lower part of the infinite channel, under gravity (see Fig. 2.1). Suppose that the streamline $A_\infty E\, D_\infty$, $\psi = 0$ consists of a horizontal line $A_\infty E$ and a free streamline ED_∞, where A_∞ is an upstream infinity and D_∞ a downstream infinity.

For convenience, we choose the origin at E, the x-axis from left to right, and y-axis upward.

Suppose q_1, p_1 and q_E, p_E are the speed and pressure at A_∞ and E respectively. We define the Froude number

$$F^2 = \frac{q_1^{\,2}}{gy_1}, \tag{2.15}$$

and the excess pressure coefficient

$$C_p = \frac{2(p_1 - p_E)}{\rho q_1^{\,2}}, \tag{2.16}$$

where y_1 is some characteristic length and ρ is the density of the fluid.

Applying Bernoulli's equation (2.1) to the two points A_∞ and E, we have

$$\frac{p_1}{\rho} + \frac{1}{2} q_1^{\,2} = \frac{p_E}{\rho} + \frac{1}{2} q_E^{\,2}. \tag{2.17}$$

Along the free stream ED_∞, we obtain

$$\frac{1}{2} q^2 + gy = \frac{1}{2} q_E^{\,2} \tag{2.18}$$

This equation (2.18) is the free surface condition. Using

(2.15) and (2.16), (2.17) and (2.18) become

$$C_p = \frac{q_E^2}{q_1^2} - 1 , \tag{2.19}$$

and
$$\frac{q}{q_1} = \left(\frac{q_E^2}{q_1^2} - \frac{2}{F^2} \frac{y}{y_1} \right)^{\frac{1}{2}} , \tag{2.20}$$

Let $z = x + iy$ and $W = \phi + i\psi$, then $\frac{dW}{dz} = u - iv = qe^{-i\theta}$.

Now, we introduce the dimensionless variables,

$$z' = \frac{z}{y_1} , \; q' = \frac{q}{q_1} \text{ and } W' = \frac{W}{\psi_1} . \tag{2.21}$$

Then

$$\frac{dW'}{dz'} = \frac{y_1}{\psi_1} \frac{dW}{dz} = \frac{q}{q_1} e^{-i\theta} = q' e^{-i\theta} .$$

Define $\zeta = \dfrac{dW'}{dz'}$, (2.22)

and $\omega = \ln\zeta = \ln q' + i(-\theta)$. (2.23)

Equation (2.22) is the normalized complex velocity and the ζ-plane is called the hodograph plane.

Equations (2.19) and (2.20) can be written in the dimensionless forms

$$C_p = {q'}_E^2 - 1 , \tag{2.24}$$

$$q' = [{q'}_E^2 - \frac{2}{F^2} y']^{\frac{1}{2}} \tag{2.25}$$

Note that (2.25) is the free surface condition in dimensionless variables.

Also, from (2.22), we have

$$z' = \int dz' = \int \frac{dW'}{\zeta} = \int e^{-\omega}\, dW' \tag{2.26}$$

Note that the fluid region in the W'-plane is an infinite strip of width unity, with free surface on $\psi' = 0$. It may be considered as a polygon with two vertices at infinity. If the two functions W' and ω in (2.26) can be expressed as functions of the same single variable t, then the integral in (2.26) may be evaluated. For the first part of the problem, to express W' as a function of t, we make use of the Schwartz-Christoffel transformation which, through explicit mappings on an upper half-plane (the t-plane) gives W' in parametric form:

$$W' = W'(t)$$

In general, the Schwartz-Christoffel transformation (see Churchill [7]) is of the form,

$$\frac{dW'}{dt} = A \prod_i (t - t_i)^{\frac{\alpha_i}{\pi} - 1} , \tag{2.27}$$

Where t_i is the t-plane coordinate related to a vertex of the polygon, and α_i the corresponding internal angle in the W'-plane; A is a constant.

The conformal mapping (2.27) maps the W'-plane onto the t-plane so that the fluid region in the W'-plane corresponds to the upper half, Im(t) > 0 of the t-plane, and the boundary of the fluid region to the real axis.

We determine W' as a function of the (half-plane) variable t by choosing a suitable conformal mapping. We see that the argument of the normalized complex velocity is known on the fixed boundaries while its magnitude is known along the free surfaces. Then, applying the Riemann-Hilbert solution to a mixed boundary value problem, we can express the function ω explicitly as a function of t.

The general solution of the Riemann-Hilbert problem in the upper half-plane is well known, for example, see Larock and Street [27], Song [46], Muskhelishvili [36] and Cheng and Rott [6]. If the imaginary part of an analytic function $Q(t)$ is known along $\mathrm{Im}(t) = 0$, the real axis of the t-plane, then the Riemann-Hilbert problem for $Q(t)$ is given by,

$$Q(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}[Q(\zeta)]d\zeta}{\zeta - t} + \sum_{j=0}^{\infty} A_j t^j , \tag{2.28}$$

where A_j are real constants and the solution is valid for all t in the upper half-plane.

The conformal mapping (2.27) maps the boundary of fluid region in the W'-plane onto the boundary, the real-axis, $\mathrm{Im}(t) = 0$ of the t-plane. Next we try to relate $\omega(t)$ to $Q(t)$ so that the imaginary part of $Q(t)$ is known at every point on $\mathrm{Im}(t) = 0$. We know either the real or imaginary part of $\omega(t)$ on $\mathrm{Im}(t) = 0$. Hence we have to construct an auxiliary function $H(t)$ which makes the imaginary part of the quotient $Q(t) = \frac{\omega(t)}{H(t)}$ known on the entire real-axis.

The general solution for $H(t)$ is

$$H(t) = \prod_i (t - b_i)^{\pm\frac{1}{2}} , \tag{2.29}$$

where b_i are real constants.

Song [46] has shown that the final solution is independent of a particular choice of $H(t)$. A branch cut is selected on the real axis to ensure that $H(t)$ is single -valued. Hence $\omega(t)$ can be constructed explicitly by means of the Riemann-Hilbert solution, where $\omega(t) = H(t)Q(t)$.

In our problem, the channel jet under gravity (see Fig. 2.2), we choose $b_1 = -1$ and $b_2 = 0$ in equation (2.29). Note that [-1,0] on the real-axis in the t-plane corresponds to the free streamline ED_∞ in ~~the~~ physical plane. The coefficients A_j, $j = 0, 1, 2, \ldots$ in (2.28) are all zero when the upstream boundary condition is applied. Hence, in this case we find

$$H(t) = \sqrt{t(t+1)}\,, \tag{2.30}$$

$$Q(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}[Q(\zeta)]}{\zeta - t}\, d\zeta\,, \tag{2.31}$$

and
$$\omega(t) = \frac{H(t)}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}[Q(\zeta)]}{\zeta - t}\, d\zeta \tag{2.32}$$

The integrand $\mathrm{Im}[Q(t)]$ is given by known constants in some parts of $\mathrm{Im}(t) = 0$, and is expressed by some (unknown) functions in other parts. Hence, it may be considered as a known quantity on the entire real-axis and the integral can then be found. The solution $\omega(t)$ is expressed implicitly in (2.32) hence a numerical method needs to be applied. In particular, some approximate formulas should be developed for the three kinds of singularities occurring in the integrand. Finally, the shape (x', y') of the free streamline ED_∞ can then be calculated from (2.26) and (2.32).

In the next Chapter, we develop a numerical method for solving this problem. In a later Chapter, we will discuss the problem in detail.

CHAPTER III

NUMERICAL METHODS

In the previous chapter, we discussed the general theory of the two methods, the Cauchy integral equation method, and the Riemann-Hilbert method for solving some two dimensional free surface problems. Since both methods involve singular integral equations, some basic quadrature formulas will be used and some approximate formulas for singularities need to be developed. First, we review some basic quadrature formulas.

QUADRATURE FORMULAS

Let f(x) be a continuous function on [0,1], and possess the necessary number of derivatives. Let h be the step size and n the number of points. Then $h = \frac{1}{n-1}$. The trapezodial, Simpson's and Simpson's three-eights are:

$$\int_0^h f(x)\,dx = h[f_0 + f_1] + 0(h^3), \tag{3.1}$$

$$\int_0^{2h} f(x)\,dx = \frac{h}{3}[f_0 + 4f_1 + f_2] + 0(h^5), \tag{3.2}$$

$$\int_0^{3h} f(x)\,dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3] + 0(h^5), \tag{3.3}$$

where $f_i = f(ih)$.

Gregory's formulas for order 1 and 4 are

$$\int_0^1 f(x)\ dx = \frac{h}{2}[f_0 + 2f_1 + \ldots + 2f_{n-2} + f_{n-1}]$$

$$+ \frac{h^2}{12}(f_o' - f_{n-1}') + 0(h^4), \qquad (3.4)$$

and

$$\int_0^1 f(x) dx = \frac{h}{2}[f_0 + 2f_1 + \ldots + 2f_{n-2} + f_{n-1}]$$

$$- h\{\frac{1}{12}(\nabla f_{n-1} - \Delta f_0) + \frac{1}{24}(\nabla^2 f_{n-1} + \Delta^2 f_0)$$

$$+ \frac{19}{720}(\nabla^3 f_{n-1} - \Delta^3 f_0) + \frac{3}{160}(\nabla^4 f_{n-1} + \Delta^4 f_0)$$

$$+ \frac{863}{60480}(\nabla^5 f_{n-1} - \Delta^5 f_0) + \frac{275}{24192}(\nabla^6 f_{n-1} + \Delta^6 f_0)]$$

$$+ 0(h^8), \qquad (3.5)$$

respectively.

For details, see Fröberg [13]. We may use the above formulas for $f(t)$ on $[a,b]$ by using a linear transformation $t = a + (b - a) x$.

3.1 Numerical Solution for Cauchy Integral Equation

We recall that the solution of a free surface problem may be expressed by equations (2.13) and (2.14), subject to the free surface condition (2.5). It was mentioned in Chapter 2 that we have to apply a numerical method to solve the problem.

Let Γ be the rectangle ABCD in the W-plane shown in Fig. 3.1. Divide each side of the rectangle into a number of subintervals. Then (2.13) and (2.14) can be written as the following discrete sums:

$$\left\{\begin{aligned} \pi U(\phi_0,\psi_0) &= -\sum_j \int_j \frac{\partial V(\phi,\psi)}{\partial s}\,\ell n\rho\,ds + \sum_j \int_j U(\phi,\psi)\frac{\partial\alpha}{\partial s}\,ds, \\ \pi V(\phi_0,\psi_0) &= \sum_j \int_j \frac{\partial U(\phi,\psi)}{\partial s}\,\ell n\rho\,ds + \sum_j \int_j V(\phi,\psi)\frac{\partial\alpha}{\partial s}\,ds, \end{aligned}\right. \tag{3.6}$$

when (ϕ_0,ψ_0) is not a corner point, and

$$\left\{\begin{aligned} \frac{\pi}{2}U(\phi_0,\psi_0) &= -\sum_j \int_j \frac{\partial V(\phi,\psi)}{\partial s}\,\ell n\rho\,ds + \sum_j \int_j U(\phi,\psi)\frac{\partial\alpha}{\partial s}\,ds, \\ \frac{\pi}{2}V(\phi_0,\psi_0) &= \sum_j \int_j \frac{\partial U(\phi,\psi)}{\partial s}\,\ell n\rho\,ds + \sum_j \int_j V(\phi,\psi)\frac{\partial\alpha}{\partial s}\,ds, \end{aligned}\right. \tag{3.7}$$

for a corner point (ϕ_0,ψ_0),

where the sums in (3.6) and (3.7) are range over all subintervals of Γ.

Note that in (3.6) and (3.7), we replace x by ϕ, v by ψ, ρ^2 by $(\phi - \phi_0)^2 + (\psi - \psi_0)^2$, tan α by $\frac{\psi-\psi}{\phi-\phi_0}$, s by ϕ or ψ, and Γ by the rectangle ABCD. We approximate each integral in both equations by Simpson's formula. Symm [52] assumed that U, V, $\frac{\partial U}{\partial s}$ and $\frac{\partial V}{\partial s}$ are constants in each subinterval and approximated integrals with the remaining integrand. Hence the results Symm [52] obtained are only exact when U and V are constants; that is, it is a first order approximation.

Let h_x and h_y be the lengths of the subintervals along the horizontal and vertical sides respectively. For each particular node, $W_0 = \phi_0 + i\psi_0$, one of the end points of the subintervals, we have to compute the values of $\ln\rho$ and $\frac{\partial\alpha}{\partial s}$, where $\rho^2 = (\phi - \phi_0)^2 + (\psi - \psi_0)^2$ and $\tan\alpha = \frac{\psi - \psi_0}{\phi - \phi_0}$ with $\phi + i\psi \neq \phi_0 + i\psi_0$.

Note that if $\phi + i\psi \neq \phi_0 + i\psi_0$, then

$$\frac{\partial\alpha}{\partial s} = \frac{\partial\alpha}{\partial\phi} = \frac{\psi - \psi_0}{(\phi - \phi_0)^2 + (\psi - \psi_0)^2},$$

when $\phi + i\psi$ is a point on a horizontal side, and

$$\frac{\partial\alpha}{\partial s} = \frac{\partial\alpha}{\partial\psi} = \frac{-(\phi - \phi_0)}{(\phi - \phi_0)^2 + (\psi - \psi_0)^2},$$

when $\phi + i\psi$ is a point on a vertical side.

Consider equation (2.11) (equation (10) of Lauck [28]) and equation (2.12) or (2.13) (equation (18) of Jaswon [20]). When ρ is small, $\frac{1}{\rho}$ is much larger than $-\ln\rho$. Hence, we can expect to obtain greater accuracy using (2.13) instead of (2.11). However, we still need a particular formula for integration

when the integral contains the singular point $W_0 = \phi_0 + i\psi_0$.
Using Taylor series expansion, we have

$$\int_{-h}^{h} \frac{\partial U(\phi,\psi)}{\partial \phi} \ln|\phi| \, d\phi$$

$$= \int_{-h}^{h} \left[\left(\frac{\partial U}{\partial \phi}\right)_0 + \phi \left(\frac{\partial^2 U}{\partial \phi^2}\right)_0 + \frac{\phi^2}{2} \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 + \frac{\phi^3}{6} \left(\frac{\partial^4 U}{\partial \phi^4}\right)_0 + \ldots \right] \ln|\phi| \, d\phi$$

$$= 2h(\ln h - 1) \left(\frac{\partial U}{\partial \phi}\right)_0 + \frac{h^3}{9} (3\ln h - 1) \left(\frac{\partial^3 U}{\partial \phi^3}\right) + 0 \, (h^5 (5\ln h - 1)), \tag{3.8}$$

where the subscript 0 refers to the point (ϕ_0, ψ_0).

Suppose that $W_0 = \phi_0 + i\psi_0$ is a point near a corner. Say $B = (\phi_0 + h_x) + i\psi_0$ in Fig. 3.1, on side AB, and we wish to evaluate $\int_B^{B+h_y} U(\phi_0 + h_x, \psi) \frac{\partial \alpha}{\partial \psi} d\psi$ in the second integral of (3.6). If h_x and h_y are small, it appears that $\frac{\partial \alpha}{\partial \psi}$ will increase rapidly as the point $\phi + i\psi$ approaches to B along CB and attains its maximum value at B. Hence we need a more accurate formula for that integral. Applying the Taylor series expansion for $U(\phi_0 + h, \psi)$ at B, we obtain

$$\int_B^{B+h_y} U(\phi_0 + h_x, \psi) \frac{\partial \alpha}{\partial \psi} d\psi$$

$$= \int_0^{h_y} \left[(U)_B + \frac{\psi^2}{2!} \left(\frac{\partial^2 U}{\partial \psi^2}\right)_B + \frac{\psi^4}{4!} \left(\frac{\partial^4 U}{\partial \psi^4}\right)_B + \frac{\psi^6}{6!} \left(\frac{\partial^6 U}{\partial \psi^6}\right)_B + \ldots \right] \cdot$$

$$\frac{h_x}{h_x^2 + h_y^2} \, d\psi$$

$$= \alpha_1 + (2r - r^2 \alpha_1) \frac{h_v^2}{2!} \left(\frac{\partial^2 U}{\partial \psi^2}\right)_B + \left(\frac{8}{3}r - 2r^3 + r^4 \alpha_1\right)$$

$$\frac{h_v^4}{4!} \left(\frac{\partial^4 U}{\partial \psi^4}\right)_B + \left(\frac{32}{5}r - \frac{8}{3}r^3 + 2r^5 - r^6 \alpha_1\right) \frac{h_v^6}{6!} \left(\frac{\partial^6 U}{\partial \psi^6}\right)_B + \cdots \quad (3.9)$$

where $r = \frac{h_x}{h_y}$, $\tan \alpha_1 = \frac{2}{r}$ and $\left(\frac{\partial^{2n+1} U}{\partial \psi^{2n+1}}\right)_B$, $n = 0, 1, 2, \ldots$

are zero.

As we are going to compute $\frac{\partial U}{\partial s}$, $\frac{\partial^2 U}{\partial s^2}$, $\frac{\partial^3 U}{\partial s^3}$ and so on, we need the values of U and V along AB, BC, CD and DA, and also outside the rectangle ABCD (see Fig. 3.2). The values of U and V on four sides of the rectangle ABCD are known either from the boundary conditions or from equations (3.6), (3.7) and (2.5). On AA', AA", BB' and so on, we may apply the principle of reflection or the reflection across the free streamline (see Lewy [31]) or some other physical reason to obtain the necessary values of U and V. If the values of U and V on some parts of AA', AA", BB', etc. cannot be determined, we must use one-side derivatives or non-symmetric formulas. Using Taylor series expansion, we obtain

$$\begin{cases} 60 h_x \left(\frac{\partial U}{\partial \phi}\right)_0 = 45 (U_1 - U_{-1}) - 9(U_2 - U_{-2}) + (U_3 - U_{-3}) + 0(h^7), \\ 8 h_x^3 \left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -13(U_1 - U_{-1}) + 8(U_2 - U_{-2}) - (U_3 - U_{-3}) + 0(h^7), \end{cases} \quad (3.10)$$

$$\begin{cases} 60h_x\left(\frac{\partial U}{\partial \phi}\right)_0 = -147U_0+360U_1-450U_2+400U_3-225U_4+72U_5-10U_6+0(h^7), \\ 8h_x^3\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -49U_0+232U_1-461U_2+496U_3-307U_4+104U_5-15U_6+0(h^7), \end{cases} \tag{3.11}$$

$$\begin{cases} 60h_x\left(\frac{\partial U}{\partial \phi}\right)_0 = -10U_{-1}-77U_0+150U_1-100U_2+50U_3-15U_4+2U_5+0(h^7), \\ 8h_x^3\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -15U_{-1}+56U_0-83U_1+64U_2-29U_3+8U_4-U_5+0(h^7), \end{cases} \tag{3.12}$$

and

$$\begin{cases} 60h_x\left(\frac{\partial U}{\partial \phi}\right)_0 = 2U_{-2}-24U_{-1}-35U_0+80U_1-30U_2+8U_3-U_4+0(h^7), \\ 8h_x^3\left(\frac{\partial^3 U}{\partial \phi^3}\right)_0 = -U_{-2}-8U_{-1}+35U_0-48U_1+29U_2-8U_3+U_4+0(h^7), \end{cases} \tag{3.13}$$

where $U_j = U_j(\phi_0 + jh_x, \psi_0)$, and the subscript 0 refers to the point $W_0 = (\phi_0, \psi_0)$.

The same formulas can be used for $\frac{\partial U}{\partial \psi}$, $\frac{\partial V}{\partial \phi}$, $\frac{\partial V}{\partial \psi}$, etc. Note that the formulas (3.10) are symmetric, (3.12) and (3.13) are non-symmetric, and (3.11) gives one-sided derivatives. The derivatives $\left(\frac{\partial^{2n} U}{\partial \psi^{2n}}\right)_B$, $n = 1,2,3$, in (3.9) can be evaluated using the following

$$\begin{cases} 180 \times \frac{h_y^2}{2!}\left(\frac{\partial^2 U}{\partial \psi^2}\right)_B = 270U_1-27U_2+2U_3-245U_0+0(h_y^{14}), \\ 72 \times \frac{h_y^4}{4!}\left(\frac{\partial^4 U}{\partial \psi^4}\right)_B = -39U_1+12U_2-U_3+28U_0+0(h_y^{14}), \\ 360 \times \frac{h_y^6}{6!}\left(\frac{\partial^6 U}{\partial \psi^6}\right)_B = 15U_1-6U_2+U_3-10U_0+0(h_y^{14}). \end{cases} \tag{3.14}$$

We compute the integral in (2.5), the free surface condition, by using (3.4) for small n (n = 2,3,4) and (3.5) for large n (n > 4).

Initial values of U and V along the sides of rectangle ABCD were estimated, using smooth piecewise quadratic interpolation, remembering that, from the boundary conditions, θ = constant and, or q = constant along some parts of the sides. For the iterative method, the Gauss-Seidel procedure was used. New values of U and V on four sides (except those which are given and U on the free streamline Γ_0), were obtained, using (3.6) and (3.7). The necessary derivatives of U and V were evaluated using (3.10) or (3.11) - (3.13). The free surface condition then provided a new value of (dimensionless) g, since the current V was known on the free streamline Γ_0, and q and ϕ on both end points of Γ_0 are also prescribed, in (2.5). Knowing g, (2.5) was then used to find U along Γ_0 also, using (3.4) and (3.5), and the derivatives of U on Γ_0 by means of (3.10).

The procedure was repeated until successive approximations differed by a prescribed small number 10^{-k}, where k = 6 in most cases, and k = 4 in a few cases. At a particular (boundary) point (ϕ,ψ), we computed U or V using (3.6) or (3.7). For convenience, we define an "iteration" to be the computation of U and V on four sides (except those which are given and U on the free streamline Γ_0); and a "cycle" to be three iterations together with evaluation of all necessary derivatives of U or V using (3.10) or (3.11) - (3.13), computing U along Γ_0 using the free surface condition (2.5), and computing the necessary derivatives of U using (3.10). In the later chapters, we shall discuss the number of cycles necessary for convergence, for both water wave problem and the jet problem.

3.2 Numerical Solution of the Riemann-Hilbert Problem

In Chapter 2, we mentioned that the shape (x', y') of the free streamline ED_∞ (for the channel jet problem) can be calculated from (2.26) and (2.32). These equations for x' and y' are singular integral equations. In our problems, we have three types of singularities, we will discuss how to overcome these difficulties in the later chapters. For those integrals which are not singular, we use the ordinary quadrature formulas which were discussed above.

Note that the equations for x' and y' along the free surface Γ_0 depend also on the values of y' along Γ_0. Therefore, we have to solve the problem numerically. We shall find that an iterative method is most suitable for our problems, and we use the Gauss-Seidel method. However, we still need an initial estimate of the solution, which we choose as the case $g = 0$, that is, the non-gravity case when the Froude number $F^2 = \infty$.

We compute x' and y' for a finite number of points. For the interpolation formula when needed, we use Lagrangian cubic interpolation formula (for equally spaced points), that is,

$$P(x) = -\frac{1}{6}(x-2)(x-3)(x-4)P_1+\frac{1}{2}(x-1)(x-3)(x-4)P_2-\frac{1}{2}(x-1)(x-2)(x-4)P_3$$
$$+\frac{1}{6}(x-1)(x-2)(x-3)P_4 , \tag{3.15}$$

where $P_i = P(i)$, $i = 1, 2, 3, 4$.

This formula is also used for computing some other functions near a singular point.

The details of the numerical solution for each individual problem will be discussed in the later chapters.

CHAPTER IV

THE STEADY WATER WAVE PROBLEM

4.1 Formulation of the Problem

Consider a symmetrical two dimensional periodic wave moving from right to left with constant velocity c on the surface of fluid of finite depth. If we superpose a constant velocity c on the fluid from left to right, the motion becomes steady, and the motion of the fluid is from left to right. The fluid is assumed to be inviscid and incompressible, and the motion irrotational. The bottom of the fluid is assumed to be horizontal, and the depth from the undisturbed water level (the mean depth of the fluid) is h. We shall investigate the motion of the fluid contained in a half-wavelength, that is, the region ABCD (see Fig. 4.1). Let the origin of the space coordinates xy be in a trough (that is, the point D in Fig. 4.1); the x-axis from left to right; and the y-axis upward. The wavelength is denoted by λ and the amplitude of the wave by a. Let the least depth of the fluid be h_1 and the mean elevation of the wave from the trough be d; then $h = h_1 + d$. The wave speed c may be defined as

$$c = \frac{2}{\lambda} \int_0^{\lambda/2} u(x,y)\, dx \tag{4.1}$$

This is Stokes' second definition of the wave speed c. For more details of discussion of this, the reader should consult

Stokes [49] or Wehausen and Laitone [55, pp. 456-7].

Recall that ϕ is the velocity potential and ψ the stream function. Let $W = \phi + i\psi$ and $z = x + iy$. Then $\frac{dW}{dz} = u - iv = qe^{-i\theta}$. Now we map the fluid region ABCD in the z-plane onto ABCD in the W-plane. All variables have been normalized in equation (2.4). Then the integral form of free surface condition in these dimensionless variables is given by equation (2.5), where q_0 is the fluid speed at the origin D. In dimensionless forms, $q_o = 1$ and $\psi_1 = 1$.

If all q's along AB, BC, CD and DA, and θ along CD have been computed, then the form of the free surface (x,y), the wave speed c, the wavelength λ, the amplitude a, the mean elevation d, the height h_1, the mean depth $\hat{h}$ and other quantities can be calculated. Here ϕ_1 and q_1, the fluid speed at the crest C, are assumed to be given. The two equations in (2.6) for computing x and y along the free surface DC are

$$\begin{cases} x(\phi) = \displaystyle\int_0^{\phi} \frac{\cos\theta(t)dt}{q(t)} \quad \text{along DC} \\ y(\phi) = \displaystyle\int_0^{\phi} \frac{\sin\theta(t)dt}{q(t)} \quad \text{along DC} \end{cases} \tag{4.2}$$

Thus

$$\begin{cases} \lambda = 2x_C , \\ a = y_C, \text{ since } x_D = y_D = 0 . \end{cases} \tag{4.3}$$

For the wave speed c,

$$c = \frac{2}{\lambda}\int_0^{\frac{\lambda}{2}} u(x,y)\ dx = \frac{2\phi_1}{\lambda} . \tag{4.4}$$

Let T_1 be the time required for a fluid particle to travel from D to C along the free surface DC. The mass transport velocity u_0 along DC is defined by

$$u_0 = c - \frac{\lambda}{2T_1} , \tag{4.5}$$

(see Ursell [54] or Longuet-Higgins [32]).

Moreover T_1 can be expressed in the form

$$T_1 = \int_0^{T_1} dt$$

$$= \int_0^{\phi_1} \frac{1}{q(\phi)^2}\ d\phi \ , \text{ along DC} . \tag{4.6}$$

The mean elevation d of the wave above the trough is defined by

$$d = \frac{2}{\lambda}\int_D^C y(x)\ dx$$

Levi-Civita [30] has shown that

$$c_1^{\ 2} + 2gd = q_0^{\ 2} \tag{4.7}$$

where

$$c_1^{\ 2} = \frac{2}{\lambda}\int_A^B u^2(x,y)dx$$

$$= \frac{2}{\lambda}\int_0^{\phi_1} q(\phi)\ d\phi \ , \text{ along AB} . \tag{4.8}$$

The depth of the fluid at the trough D is

$$h_1 = \int_{-h_1}^{0} dy$$

$$= \int_{0}^{1} \frac{1}{q(\psi)}\, d\psi \qquad \text{along AD} . \tag{4.9}$$

4.2 Numerical Solution

Let Γ be the rectangle ABCD in the W-plane in Fig. 4.1, and ϕ_1 and q_1 be known quantities. Divide each side of the rectangle into a number of equal subintervals. Let h_x and h_y be the lengths of the subintervals along the horizontal and vertical sides respectively. The two functions U and V are $\ln q(\phi,\psi)$ and $-\theta(\phi,\psi)$. Note that $\theta = 0$ along AD, AB and BC of the rectangle ABCD. In terms of U and V, V = 0 along these three sides.

As was mentioned in Chapter 3, we need the values of U and V outside the rectangle ABCD (see Fig. 3.2). Since V = 0 on AD, AB and BC, we can apply the principal of reflection to obtain the necessary values of U along AA', AA", BB', BB", CC' and DD', and V along CC' and DD'. For values of U on CC" and DD", we use the reflection principle across the free surface (see Appendix A). From Appendix A, we find

$$\frac{dG}{d\psi} = \frac{1}{\frac{dF}{d\psi}\, 8q\, [i(G-F)-h_y]} \,, \tag{4.10}$$

$$\begin{aligned} \text{where} \quad & F(0,\psi) = x(0,\psi) + iy(0,\psi), \\ \text{and} \quad & F(0,\psi) = \overline{G}(0,-\psi),\ \psi \geq 0 \,. \end{aligned} \tag{4.11}$$

When applying (4.10) on AD and BC, that is $\phi = 0$ and $\phi = \phi_1$, we obtain the values of U along CC" and DD", since in this case, x = constant and hence $\frac{dF}{d\psi} = u = U$.

We estimate the initial values of U and V on the sides of ABCD and start the Gauss-Seidel iterative procedure. We obtain the new values of U on AD, AB and BC, and V on DC using (3.6), and U at A and B (corners) using (3.7). Then we evaluate all the necessary derivatives of U and V using (3.10) and (3.11) - (3.13). Using free surface condition (2.5), we obtain a new (dimensionless) g, since the current V is known on DC, and $q = q_1$ and $\phi = \phi_1$ are given in (2.5). Knowing g, we compute U along DC also, using (2.5), and evaluate the necessary derivatives of U using (3.10). We repeat the procedure for N (e.g. N = 20 or 30) cycles, or until the successive approximations differed by a prescribed small number 10^{-k}, where k = 6 in most cases, but k = 4 in a few.

4.3 Numerical Results

It was mentioned, in section 4.1, that we had normalized so that the flux $\psi_1 = 1$, and the fluid speed at the trough $q_0 = 1$. We may then expect the iterative procedure outlined above to work well in the vicinity of $\phi_1 = 1$ and when q_1 is not too close to zero. When ϕ_1 is large, shallow water theory can be applied, and when ϕ_1 is small, deep water theory. When $q_1 = 1$, uniform flow is obtained, and as $q_1 \to 0$, the highest wave is approached.

It was found most convenient, for our purpose, to fix ϕ_1 and q_1, and then compute a, h, λ, c^2 and g. For this reason, it was not easy to produce results which could be compared with those of Thomas [53], which could require selection of the values of ϕ_1 and q_1, so that resultant values of $\frac{h}{\lambda}$ agreed with those used by Thomas. However, this was done, if at the expense of extra computing time, and the results are shown in Fig. 4.2. It was observed, from the computed results, that ϕ_1 is a decreasing function of $\frac{h}{\lambda}$ for fixed q_1. This made it easier to adjust ϕ_1, when necessary to produce a desired value of $\frac{h}{\lambda}$.

Two particular difficulties arose in the computing procedure adopted. The first was that the integrand of the second integral of (3.6) changed very rapidly for points near a corner, because of the form of $\frac{\partial\alpha}{\partial\psi}$. This difficulty was circumvented by using a formula (3.9), yielding greater accuracy. Secondly, the first integral of (3.6) became singular when the j^{th} subinterval contained 0. This, in turn, was overcome by expanding $\frac{\partial U}{\partial\psi}$ as MacLaurin series and integrating term by term, as shown in (3.8).

It was found that the lengths h_x and h_y, the subintervals, had to be comparable, in order to obtain uniform accuracy around the contour. When considering shallow, or deep, water waves, it was found that it was not possible to obtain the same rapidity of convergence. However, we were principally interested in obtaining the profiles of waves in which the wavelength and depth were of the same order of magnitude.

The computed results are summarized in Tables 4.1 and 4.2. Between three and four sets of values of q_1, ϕ_1, $\frac{c^2}{gh}$ and $\frac{a}{h}$ have been found for each of five values of $\frac{h}{\lambda}$. These data have been plotted in Fig. 4.2 and compared with those of Thomas. When $0.15 < \frac{h}{\lambda} < 0.6$, the results agreed well, except for $\frac{h}{\lambda} = 0.15$ and small values of q_1. When $\frac{h}{\lambda} = 0.10632$, the two curves are quite different. When $\frac{h}{\lambda} > 0.6$, there are no corresponding values in Thomas' paper. No tables have been given by Schwartz [42], or Wehausen and Laitone [55], but a comparison with their graphs, when $0.0618 < \frac{h}{\lambda} < 0.3$ and 0.05

$< \frac{h}{\lambda} < 0.6$ respectively, shows good agreement, producing smooth and comparable curves. Table 4.3 shows the velocity of the shape (x,y) of the free surface DC when $\phi_1 = 0.83913$ and $q_1 = 0.6$.

The iterative procedure was terminated when successive approximations agreed up to sixth decimal place, which in most cases, required about 24 cycles. When $\frac{h}{\lambda} = 0.10632$, 36 cycles were required to obtain agreement up to the fourth decimal place.

It was found that the difference between continuations obtained using (4.10) (Lewy's method [31]) and those obtained using Shiffman's method [45] (i.e. $g = 0$) was quite small, and did not materially effect the shape of the wave except for an occasional change in the sixth decimal place.

4.4 Numerical Checks

A number of checks were applied to the numerical procedure. First, the values of v obtained by using (2.2) and (2.6) were compared, and in most cases, were found to agree up to the seventh decimal place. Secondly, a number of trends were observed, assuming, as we did, that the streamlines crossing AD are concave up, and those crossing BC are concave down, then q should decrease along the free surface from trough to crest, and around the contour DABC, obtaining its maximum at D and minimum at C. θ should rise to a single maximum between D and C. Thirdly, and most importantly, the total momentum flux across AD should be equal to that across BC. This result can easily be established using the following analogy of a lemma of Levi-Civita's [30, p. 276]:

$$\int_\Gamma q^2\, dy = 2 \int_\Gamma u\, (udy - vdx) , \tag{4.12}$$

where $q^2 = u^2 + v^2$, and u and v are conjugate harmonic functions. The lemma follows at once from Green's Theorem.

When Γ is a contour ABCD in the xy-plane (see Fig. 4.1), we find that (in dimensionless coordinates):

$$M_{AD} \stackrel{\text{def}}{=\!=} \int_A^D (p + u^2)\, dy = \frac{1}{2} \int_A^D u^2 dy + h_1(q_0{}^2 + gh_1) ,$$

where $p + \frac{1}{2}(u^2 + v^2) + gy = \frac{1}{2} q_0{}^2 + gh_1 = \frac{1}{2} q_1{}^2 + gh_2$

and $h_2 = h_1 + a$ is the depth BC.

Likewise,

$$M_{BC} \overset{\text{def}}{=\!=} \int_B^C (p + u^2)\, dy = \frac{1}{2} \int_B^C u^2 dy + h_2 (q_1{}^2 + gh_2) .$$

Applying (4.12) it is easy to show that $M_{AD} = M_{BC}$, that is, the total momentum flux across AD is equal to that across BC.

Sample checks yielded the following comparison (in dimensionless form):

q_1	ϕ_1	M_{AD}	M_{BC}
0.6	0.83913	0.1002764	0.1002297
0.6	1.68860	0.1914649	0.1914360

CHAPTER V

VERTICAL JET FROM A RECTANGULAR VESSEL

5.1 Formulation of the Problem

Consider a large vessel (see Fig. 5.1a) filled with water, with an aperture located at the centre of its bottom. A jet issues downward from the aperture under gravity, with two free surfaces. Assume that the height of the vessel is h_1 and the half-width of the aperture b. The flow is assumed to be steady, two-dimensional, inviscid and incompressible, and the motion irrotational. The fluid motion is symmetric with respect to the centre line BC_∞, and hence only half of the fluid region has to be considered (see Fig. 5.1b). It has been shown by Carter [5] that the free streamlines should be asymptotic to this vertical line BC_∞. From conservation of mass, the velocity far downstream from the aperture approaches infinity. We choose the origin of the xy-plane at the point E, the x-axis from left to right and the y-axis upward. Assume that the water on the top AB of the vessel is undisturbed, and the velocity along it constant and equal to q_1. Let x_1 be the half-width of the vessel. Furthermore, we assume the velocity along DC is constant and equal to q_2, where D is a point some distance from the origin E and DC is a horizontal line. Let the length of DC be x_2.

Recall that ϕ is the velocity potential and ψ the stream function. Let $W = \phi + i\psi$ and $z = x + iy$. Then $\frac{dW}{dz} = qe^{-i\theta} = u - iv$. Now we map the fluid region ABCDEF in the z-plane onto the rectangle ABCDEF in the W-plane. All variables have been normalized in equation (2.4). Note that q_0 is the fluid speed at the origin E, and $\phi_1 = \phi_2 + \phi_3$ (see Fig. 5.2). The integral form of the free surface condition in these dimensionless variables is given by equation (2.5). In dimensionless forms, $q_0 = 1$ and $\psi_1 = 1$.

If all q's along four sides of the rectangle ABCDEF, and θ along the free surface ED, have been computed, then the form of the free surface (x,y), the heights h, h_1 and h_2 (see Fig. 5.1b) the half width b of the aperture, the half-width x_1 of the vessel, the length x_2 of DC, and other quantities may be computed. Here ϕ_1, q_1, the velocity along AB, and q_2, the velocity along DC, are assumed to be given. The two equations in (2.6) for computing x and y along the free surface are

$$\begin{cases} x(\phi) = \int_0^{\phi} \frac{\cos\theta(t)dt}{q} & \text{along ED} \\ y(\phi) = \int_0^{\phi} \frac{\sin\theta(t)dt}{q} & \text{along ED} \end{cases} \tag{5.1}$$

Thus

$$b = x_D + x_2 \, ,$$
$$h_2 = -y_D, \quad \text{since } x_E = y_E = 0 \tag{5.2}$$

The height h (see Fig. 5.1b) can be computed from

$$h = \int_{-h_2}^{h_1} dy$$

$$= \int_0^1 \frac{1}{q(\psi)} \, d\psi \; , \quad \text{along BC} \, , \tag{5.3}$$

and hence

$$h_1 = h - h_2 . \tag{5.4}$$

5.2 Numerical Solution

Let Γ be the rectangle ABCDEF in the W-plane in Fig. 5.2, and ϕ_1, q_1 and q_2 be known quantities. Divide each horizontal side of the rectangle into a number of subintervals. Let h_x be the length of each subinterval. The two functions U and V are $u(\phi,\psi)$ and $-v(\phi,\psi)$, where u and v are components of the fluid velocity. Note that $\theta = 0$ along AB, CD and EF, $\theta = \frac{\pi}{2}$ along BC and FA, and $q = q_1$ and $q = q_2$ along AB and CD respectively. In terms of U and V, $U = 0$ on AB, BC, CD and FA, and $V = 0$, q_1 and q_2 on EF, AB, CD respectively. Then the Cauchy integral formulas (2.13) and (3.6) become

$$\pi V(\phi_0,\psi_0) = \sum_{j,FD}\int_j \frac{\partial U(\phi,\psi)}{\partial\phi}\ln\rho\, d\phi + \sum_{j,AF}\int_j V(\phi,\psi)\frac{\partial\alpha}{\partial\phi}d\phi$$

$$+ \sum_{j,ED}\int_j V(\phi,\psi)\frac{\partial\alpha}{\partial\phi}d\phi + \alpha_1 q_1 + \alpha_2 q_2 ,$$

$$\pi V(\phi_0,\psi_0) = \sum_{j,FD}\int_j \frac{\partial U(\phi,\psi)}{\partial\phi}\ln\rho\, d\phi + \sum_{j,BC}\int_j V(\phi,\psi)\frac{\partial\alpha}{\partial\phi}d\phi$$

$$+ \alpha_1 q_1 + \alpha_2 q_2 ,$$

$$\pi U(\phi_0,\psi_0) = \sum_{j,BC}\int_j \frac{\partial V(\phi,\psi)}{\partial\phi}\ln\rho\, d\phi - \sum_{j,AF}\int_j \frac{\partial V(\phi,\psi)}{\partial\phi}\ln|\phi-\phi_0|d\phi$$

$$- \sum_{j,ED}\int_j \frac{\partial V(\phi,\psi)}{\partial\phi}\ln|\phi-\phi_0|d\phi ,$$

$$\pi V(\phi_0, \psi_0) = \sum_{j,FD} \int_j \frac{\partial U(\phi,\psi)}{\partial \phi} \ln \rho \, d\phi + \sum_{j,BC} \int_j V(\phi,\psi) \frac{\partial \alpha}{\partial \phi} \, d\phi$$

$$+ \alpha_1 q_1 + \alpha_2 q_2 , \tag{5.5}$$

where $\rho^2 = 1 + (\phi - \phi_0)^2$, $\tan \alpha = \dfrac{\psi - \psi_0}{\phi - \phi_0}$, with $\phi + i\psi \neq \phi_0 + i\psi_0$, α_1 and α_2 are angles shown in Fig. 5.3, and $\sum_{j,FD}$ means the sum ranged over all subintervals on side FD; when (ϕ_0, ψ_0) is not a corner point along any one of BC, AF, FE and ED.

We compute the derivatives of U and V using (3.10), or (3.11) - (3.13). Here we use (3.10) if we know all the necessary values of U or V, otherwise (3.11) - (3.13) should be applied. We may need the values of V on AA', BB', CC' and DD' (see Fig. 5.4), and U on FF' and V on FF" (see Fig. 5.5). However, we cannot apply the principle of reflection to obtain V on AA', BB' and FF". For values of V along CC' and DD', we use the Lagrangian cubic-interpolation formula (3.15). Because of the physical meaning (see Fig. 5.5), we may apply the principle of reflection to obtain the values of U on FF'.

We compute the initial values of U and V on the sides of the rectangle ABCDEF and start the Gauss-Seidel iterative procedure. We compute new values of V on BC, FA and ED, and U on FE, using (5.5). Then we evaluate all necessary derivatives of U and V using (3.10), or (3.11) - (3.13). Using the free surface condition (2.5), we obtain a new (dimensionless) g, since the current V are known along the free surface ED, and $q = q_2$ and $\phi = \phi_2$ are given in (2.5). Knowing g, we compute U along ED also, using (2.5), and evaluate the necessary derivatives of U using (3.10). We repeat the procedure until the successive approximations differed by a prescribed small number 10^{-k}, e.g. $k = 6$.

5.3 Numerical Results and Discussions

It was mentioned above that we normalized the flux $\psi_1 = 1$ and the fluid speed at the origin E, $q_0 = 1$. We may expect the iterative procedure outlined above to work well for ϕ_1 not too large (say $\phi_1 < 10$) and q_1 not too close to 1 or not too small (say $0.0005 < q_1 < 0.02$). When ϕ_1 is large or q_1 is small, we have the jet flow with vessel height h_1 large; and q_1 close to 1, the vessel height $h_1 \to 0$, hence we may not assume the velocity on the surface AB of the vessel to be constant.

We fix ϕ_1, q_1 and q_2, then compute h_1, h_2, h, b, x_1, x_2 $\frac{x_2}{b}$ and q_{0+h}, where q_{0+h} is the fluid speed at the very first node near B on BC (see Figures 5.1 and 5.2). When $q_1 = 0.025$ and $q_2 = 1.11111$, Table 5.1 shows how ϕ_1 affects the solution. From the computed results, it shows the vessel height h_1 and the half-width of the aperture b increase as ϕ_1 increases. When $q_2 = 1.11111$ and $q_2 = 1.15$, Tables 5.2 and 5.3, respectively, show four different values of q_1 with suitably chosen values of ϕ_1, and the corresponding values of other parameters.

One particular difficulty appearing in the computing procedure is the logarithmic singularity $W_0 = \phi_0 + i\psi_0$ in (5.5) when the subinterval contains the point W_0. This was overcome by developing a formula (3.8), using Maclaurin's series and term-wise integration.

Since the velocities along AB and CD are assumed to be constant, therefore we do not have the difficulty for points near a corner as those of the water wave problem, and we do not have to compute U or V on AB and CD, as we did in the water wave problem. This saves a lot of the computing time and we may expect quicker convergence and less time compared to the solution of the water wave problem. We use 29 points on each horizontal side of the rectangle and, in most cases, it requires only 8 cycles to obtain up to three significant decimal places, and 20 cycles for 6 places.

When successive approximations (of U and V) agree up to the desired number of decimal places, we compute the shape (x,y) of the free surface ED using (5.1), and use (3.4) and (3.5) to evaluate the integrals in (5.1). Tables 5.4 - 5.11 show some such free surfaces, which are plotted in Figures 5.6 and 5.7.

CHAPTER VI

VERTICAL JET FROM AN INFINITE CHANNEL

6.1 Formulation of the Problem

Consider steady, two-dimensional flow under gravity between two horizontal planes. The fluid is assumed to be inviscid and incompressible, and the motion irrotational. When a transverse aperture is made on the lower plane, the fluid will emerge as a jet bounded by two free surfaces. The flow in the channel is formed by two uniform streams approaching from A_∞ and A'_∞ with speed q_1. Because of the symmetry, we only have to consider half of the fluid region (see Fig. 6.1a,b). For convenience, we choose E to be the origin in the z-plane, the x-axis from left to right and the y-axis upward.

Suppose that q_1, p_1 and q_E, p_E are the speed and pressure at A_∞ and E, respectively. We have defined the Froude number F^2 and the excess pressure coefficient C_p in (2.15) and (2.16) respectively. Let $z = x + iy$ and $W = \phi + i\psi$. Also, we have introduced the dimensionless variables z', q' and W' in (2.21). In dimensionless forms, C_p and the free surface condition were expressed in (2.24) and (2.25) respectively. The dimensionless physical variable z' was given in (2.26) in terms of two functions W' and ω. That is,

$$z' = \int e^{-\omega}\, dW' \tag{6.1}$$

If we can express the two functions W' and ω as functions of a single variable t, then the integral in (6.1) can be found. For the first half of the problem, to express the function W' as a function of a single variable t, we use a conformal mapping (2.27). This mapping should map the fluid region in the W'-plane (see Fig. 6.2) into the upper half-plane, the t-plane, and the boundary of the fluid region onto the real-axis, the boundary of the t-plane (see Fig. 6.3). To ensure the uniqueness of the mapping, we choose three corresponding points in the following ways,

$$\begin{cases} E: & W' = 0,\ t = -1 , \\ D_\infty: & W' = \infty,\ t = 0- , \\ A_\infty: & W' = -\infty,\ t = \infty . \end{cases} \tag{6.2}$$

The mapping is

$$W' = i - \frac{1}{\pi} \log t , \tag{6.3}$$

where the constants are chosen so that for any real $t \geq 0$ on the real-axis, $\mathrm{Im}\,(W') = \psi'_1 = 1$.

For the second half of the problem, to express ω as a function of the single variable t, we introduce the Riemann-Hilbert method for a mixed-boundary-value problem in the upper half-plane. The general solution of the Riemann-Hilbert problem for an analytic function $Q(t)$ in the upper half-plane was given in (2.28). Now, we try to relate the function $\omega(t)$ to the function $Q(t)$. From (2.28), we find that $Q(t)$ is expressed in terms of the imaginary part of Q along the real-axis, the boundary of the t-plane. Thus, we have to examine the values of $\omega(t)$ along the real-axis of the t-plane, and we find that

$$\begin{cases} \mathrm{Im}\,[\omega(t)] = 0\,, & -\infty < t < -1\,, \\ \mathrm{Re}\,[\omega(t)] = \frac{1}{2}\log\,[q'^2_E - \frac{2}{F^2}\,v'(t)]\,, & -1 < t < 0\,, \\ \mathrm{Im}\,[\omega(t)] = \frac{\pi}{2}\,, & 0 < t < t_B\,, \\ \mathrm{Im}\,[\omega(t)] = 0\,, & t_B < t < \infty\,. \end{cases} \tag{6.4}$$

This means that we know either the real or imaginary part of $\omega(t)$ along the real-axis of the t-plane. Then we must next construct an auxiliary function $H(t)$ which makes the quotient $Q(t) = \frac{\omega(t)}{H(t)}$ known on the entire real axis and always imaginary. The general form of $H(t)$ was given in (2.29). One such function $H(t)$ is

$$H(t) = \sqrt{t(t+1)} \tag{6.5}$$

We choose a branch cut (-1,0) on the real-axis to ensure the function $H(t)$ is single-valued. Using (6.4) and (6.5), we obtain

$$\mathrm{Im}[Q(t)] = \begin{cases} 0\,, & -\infty<t<-1\,, \\ -\frac{1}{2}\left\{\log\,[q'^2_E - \frac{2}{F^2}\,v'(t)]\right\}[-t(t+1)]^{-\frac{1}{2}}\,, & -1<t<0\,, \\ \frac{\pi}{2}\,[t(t+1)]^{-\frac{1}{2}}\,, & 0<t<t_B\,, \\ 0\,, & t_B<t<\infty\,. \end{cases} \tag{6.6}$$

Next, we examine the upstream condition. As we approach the point A_∞ along the lower plane, $H(t) \sim t$ and $\omega(t) \to \log q'_1 = 0$. Therefore, $Q(t) = \frac{\omega(t)}{t} \to 0$, and from (2.28), $A_j = 0$, $j = 0, 1, 2, \ldots$. Thus (2.28) becomes

$$Q(t) = \frac{1}{\pi}\int_{\infty}^{\infty} \frac{\mathrm{Im}[Q(\zeta)]d\zeta}{\zeta - t} \tag{6.7}$$

Hence, the function $\omega(t)$ is given by

$$\omega(t) = \frac{H(t)}{2\pi}\left[-\int_{-1}^{0} \frac{\log[q'^2_E - \frac{2}{F^2} y'(\zeta)]d\zeta}{(\zeta - t)\sqrt{-\zeta(\zeta+1)}} + \pi\int_0^{t_B} \frac{d\zeta}{(\zeta-t)\sqrt{\zeta(\zeta+1)}}\right]. \tag{6.8}$$

Applying the upstream condition again,

$$0 = \frac{1}{2\pi}\int_{-1}^{0} \frac{\log[q'^2_E - \frac{2}{F^2} y'(\zeta)]d\zeta}{\sqrt{-\zeta(\zeta+1)}} - \frac{1}{2}\int_0^{t_B} \frac{d\zeta}{\sqrt{\zeta(\zeta+1)}} . \tag{6.9}$$

The second term can be integrated exactly and is

$$\int_0^{t_B} \frac{d\zeta}{\sqrt{\zeta(\zeta+1)}} = \log\left| 2t_B + 1 - \sqrt{t_B^2 + t_B}\right| \tag{6.10}$$

Let

$$I_1 = \frac{1}{\pi}\int_{-1}^{0} \frac{\log[q'^2_E - \frac{2}{F^2} y'(\zeta)]d\zeta}{\sqrt{-\zeta(\zeta+1)}} \tag{6.11}$$

Then (6.9) becomes

$$I_1 = \log \left| 2\, t_B + 1 - \sqrt{t_B^{\,2} + t_B} \right| ,$$

or,

$$t_B = \frac{(1 - e^{I_1})^2}{4\, e^{I_1}} = \sinh^2 \left(\tfrac{1}{2} I_1\right) \tag{6.12}$$

Note that the integral (6.11) can be evaluated when $y'(t)$ along the free surface ED_∞ are known. Hence, t_B may be regarded as a known quantity which satisfies (6.12).

6.2 Solution of the Problem

Now, we consider the function $\omega(t)$ on $-1 < t < 0$, that is, along the free surface ED_∞. For $-1 < t < 0$, equation (6.8) will lead to

$$\omega(t) = \frac{\sqrt{-t(t+1)}}{2\pi}\left[-\int_{-1}^{0} \frac{\log[q'_E{}^2 - \frac{2}{F^2} y'(\zeta)]d\zeta}{(\zeta-t)\sqrt{-\zeta(\zeta+1)}} + \pi\int_{0}^{t_B} \frac{d\zeta}{(\zeta-t)\sqrt{\zeta(\zeta+1)}}\right] \tag{6.13}$$

Again, the second integral can be evaluated, and is

$$\int_{0}^{t_B} \frac{d\zeta}{(\zeta-t)\sqrt{\zeta(\zeta+1)}} = \sqrt{-t(t+1)}\left[\frac{\pi}{4} + \sin^{-1}\left[1 + 2t + \frac{2t(t+1)}{t_B-t}\right]\right] \tag{6.14}$$

Let

$$I_2(t) = \frac{\sqrt{-t(t+1)}}{\pi}\int_{-1}^{0} \frac{\log[q'_E{}^2 - \frac{2}{F^2} y'(\zeta)]\zeta d}{(\zeta-t)\sqrt{-\zeta(\zeta+1)}}, \tag{6.15}$$

where

$$\int_{-1}^{0} = \lim_{\varepsilon\to 0}\left[\int_{-1}^{t-\varepsilon h} + \int_{t+\varepsilon h}^{0}\right]$$ and $h > 0$ is small.

Substituting (6.14) and (6.15) into (6.13), we obtain

$$\omega(t) = i\,[J(t) + I_2(t)] + \frac{1}{2}\log[q'_E{}^2 - \frac{2}{F^2} y'(t)], \tag{6.16}$$

where $J(t) = \frac{\pi}{4} + \frac{1}{2} \sin^{-1} [1+2t+\frac{2t(t+1)}{t_B-t}]$. (6.17)

Note that the last term of (6.16) is obtained when we evaluate

$$\lim_{\varepsilon \to 0} \int_{t-\varepsilon h}^{t+\varepsilon h} \frac{\log[q'_E{}^2 - \frac{2}{F^2} v'(\zeta)] \, d\zeta}{(\zeta-t)\sqrt{-\zeta(\zeta+1)}} \tag{6.18}$$

of (6.13), in a small interval $(t-\varepsilon h, t+\varepsilon h)$ centred at the singularity t, with $h > 0$ small. We consider the non-singular part of the integrand to be constant in the interval and equal to the value of it at $\zeta = t$, so that we may integrate the remaining integrand exactly.

From (6.16) and (2.23), we find that

$$\begin{cases} q'(t) = [q'_E{}^2 - \frac{2}{F^2} v'(t)]^{\frac{1}{2}}, \\ \theta(t) = -[J(t) + I_2(t)]. \end{cases} \tag{6.19}$$

The dimensionless physical variable $z' = x' + iy'$ can be computed using (6.1) and (6.3) as follows:

$$\begin{aligned} z' = x' + iy' &= \int_0^{W'} e^{-\omega} \, dW' \\ &= -\int_{-1}^{t} \frac{\exp[i\theta(\zeta)] \, d\zeta}{\pi \zeta q'(\zeta)} \\ &= -\int_{-1}^{t} \frac{[\cos\theta(\zeta) + i \sin\theta(\zeta)] \, d\zeta}{\pi \zeta q'(\zeta)}, \end{aligned} \tag{6.20}$$

whence we obtain,

$$\begin{cases} x' = -\dfrac{1}{\pi}\displaystyle\int_{-1}^{t} \dfrac{\cos\theta(\zeta)\,d\zeta}{\zeta q'(\zeta)}, \\ \\ y' = -\dfrac{1}{\pi}\displaystyle\int_{-1}^{t} \dfrac{\sin\theta(\zeta)\,d\zeta}{\zeta q'(\zeta)} \end{cases} \qquad \text{for } -1 < t < 0 \tag{6.21}$$

In particular, the half-width b' of the aperture is given by

$$b' = -\frac{1}{\pi}\int_{-1}^{0} \frac{\cos\theta(\zeta)\,d\zeta}{\zeta q'(\zeta)} \tag{6.22}$$

6.3 Numerical Solution

The solution of the problem is given by (6.21) subject to the condition that t_B satisfies (6.12). However the integrals in both equations (6.21) and (6.12) are non-linear and depend on the unknown values of y' along the free streamline ED_∞, and hence cannot be solved explicitly. Hence numerical methods must be introduced.

In computing x' and y' in (6.21) and t_B in (6.21), we assume that all y' along the free surface ED_∞ are known. However, this is not the case. Therefore, when we choose a particular quadrature formula, we still cannot compute the solution numerically. Hence an iterative method should be applied to solve the problem.

Initial approximation is found by choosing $g = 0$, that is, the non-gravity case. In this case, the Froude number $F^2 = \infty$ and the problem is equivalent to that of Conway [9]. The integrals I_1 in (6.11) and $I_2(t)$ in (6.15) can be evaluated exactly. They are

$$\begin{cases} I_1 = \log {q'_E}^2 , \\ I_2(t) = -\dfrac{1}{2\pi} \log \left(\dfrac{1+t}{1-t}\right), \qquad -1 < t < 0 \end{cases} \tag{6.23}$$

Hence t_B in (6.12) and x' and y' in (6.21) become

$$t_B = \left(\frac{1}{2q'_E} - \frac{q'_E}{2}\right)^2 , \tag{6.24}$$

and

$$\begin{cases} x' = -\dfrac{1}{\pi q'_E} \displaystyle\int_{-1}^{t} \dfrac{\cos\theta(\zeta)d\zeta}{\zeta}, \\ y' = -\dfrac{1}{\pi q'_E} \displaystyle\int_{-1}^{t} \dfrac{\sin\theta(\zeta)d\zeta}{\zeta}, \end{cases} \tag{6.25}$$

where $\theta(t) = -[I_2(t) + J(t)]$

$$= \frac{1}{2\pi} \log \left(\frac{1+t}{1-t}\right) - \frac{\pi}{4} - \frac{1}{2} \sin^{-1} \left[1 + \frac{2t(t_B+1)}{t_B-t}\right]. \tag{6.26}$$

Note that the integrands of both equations in (6.25) are functions of t and are known. Thus the solution can be obtained explicitly.

Three types of singularities appear in the computing procedure. The first type is in the integrand (of (6.11) and (6.15)) involving a square-root singularity at one end-point of the range of integration; the second when the integrand has a simple pole within the range of integration; and the third when the integrand has a square-root singularity and a logarithmic singularity at one end-point. The method for overcoming the first two types of singularities are the same and was described in the previous section. Applying the method for $t = -1$, an end-point, we found that, the value for the integral around the singularity $t = -1$, is zero. For the third type of singularity, consider

$$I_0(t) = \lim_{\delta\to 0} \int_{-\delta}^{0} \frac{\log\,[q'^2_E - \frac{2}{F^2} v'(\zeta)]d\,\zeta}{(\zeta - t)\sqrt{-\zeta(\zeta+1)}} \tag{6.27}$$

of (6.15) in a small interval $(-\delta, 0)$ with $\delta > 0$. When $\zeta < 0$ is small, then from (2.2), (6.3) and $\frac{dW}{dz} = q\, e^{-i\theta}$, we have

$$q \sim \sqrt{-y}\ , \ (-\zeta) \sim e^{-W} \text{ and } q \sim (-\frac{dW}{dy})\ ;$$

which lead to

$$W \sim (-y)^{\frac{3}{2}} \text{ and } (-y) = [\log(\zeta)]^{\frac{2}{3}} \tag{6.28}$$

Using (6.28), (6.27) becomes

$$I_0 \sim \lim_{\delta \to 0} \int_{-\delta}^{0} \frac{\log\ [-\log(-\zeta)]d\zeta}{\sqrt{-\zeta}},$$

$$= \lim_{\delta \to 0} \int_{-\delta}^{0} \frac{\sqrt{-\zeta}\ \log\ [-\log\ (-\zeta)]d\zeta}{(-\zeta)}$$

But $\sqrt{-\zeta}\ \log\ [-\log(-\zeta)] \leq \sqrt{-\zeta}\ \log\ (-\frac{1}{\zeta}) = -\sqrt{-\zeta}\ \log\ (-\zeta) \to 0$, as $\zeta \to 0-$. Hence

$$2I_0 \sim [-\sqrt{-\zeta}\ \log\ (-\zeta)]_{\delta \to 0-} \lim_{\delta \to 0-} \int_{-\delta}^{\delta} \frac{d\ \zeta}{(-\zeta)} \tag{6.29}$$

$$= 0 \cdot \pi i = 0\ ,$$

Note that (6.29) was obtained by the same method as equation (6.18).

We may compute x' and y' in (6.21) for "all" but a finite number of t, $-1 < t < -\delta$ and $\delta > 0$, but due to computer size and costs, we only select a relatively small number $(N + 1)$ (usually $N = 20$) of points in $(-1, -\delta)$. These points are distributed equally on $(-1, -\delta)$. Then for any particular point t, x' and y' can be found immediately using Lagrangian cubic-interpolation formula (3.15). Furthermore, we only know the integrands of (6.11) and (6.15) at $(N - 1)$ points. Their values at all other points t within the range of interest can be evaluated using (3.15).

One of the three parameters ς_p, F^2 and α'_E may be prescribed. If $\varepsilon > 0$ and $\delta > 0$ are also given, with the initial approximation (take $g = 0$), we may start the iterative procedure. We use the Gauss-Seidel method. We compute t_B using (6.12), and $I_2(t)$ and $J(t)$ using (6.15) and (6.17) respectively. These values are then used to evaluate x' and y' in (6.21). We repeat the procedure, until the successive approximations differ by a prescribed small number 10^{-k}, where $k = 4$ or 6.

6.4 Numerical Results and Discussions

The two parameters $h\varepsilon > 0$ and $\delta > 0$ should be small. We choose $h = 0.05$, $\varepsilon = 0.025$ and $\delta = 0.0005$. Table 6.1 shows the relationship between F^2 and b' for several values of q'_E (and hence C_p). In Fig. 6.4, we plot the data, which show that b' is an increasing function of F^2 for a fixed q'_E (or C_p). Table 6.2 shows the relationships between q'_E, C_p, and b' for some values of F^2. In Fig. 6.5 we plot b' against C_p, and from the graph we find that C_p decreases as b' increases for a fixed F^2. Table 6.2 also shows that Conway's [9] result ($b' = 0.42$ and $F^2 = 0.55$) corresponds to approximately $C_p = 0.56$. Table 6.3 and 6.4 show how ε (or δ) affects the solution when we keep δ (or ε) fixed.

The dimensionless variables x' and y' along the free streamline ED_∞ are given in (6.21). Tables 6.5 - 6.20 show some free streamlines, which are plotted in Figs. 6.6 to 6.10 for keeping either F^2 or q'_E (and hence C_p) fixed.

After an initial guess ($g = 0$) of y' along ED_∞, we start the iterative procedure. The iterative process becomes stable after the second iteration, and in the third iteration it produces 3 significant decimal places. For 5 significant decimal places we need 6 iterations. The CPU time for each iteration was approximately 0.5 seconds using an IBM 360/65 computer.

Finally we conclude that the solution can be imoroved by using more points and/or double precision arithmetic in the computer. However both of these would involve substantial increases in time and cost.

CHAPTER VII

THE SOLITARY WAVE

7.1 Formulation of the Problem

Consider a solitary wave, that is a single elevation of invariant form, moving from right to left with constant speed q_1 on the surface of an inviscid, incompressible fluid which is at rest at infinity. The motion is assumed to be two dimensional and irrotational, the bottom horizontal and the depth of the undisturbed fluid y_1. If we superpose a constant velocity q_1 from left to right on the fluid, the motion becomes steady and the velocity at infinity becomes q_1 from left to right. For convenience, we choose the origin of the x-y plane at the point C (see Fig. 7.1); the x-axis from left to right, and the y-axis upward. The amplitude is denoted by a.

Recall that ϕ is the velocity potential and ψ the stream function. Let $W = \phi + i\psi$ and $z = x + iy$. Then $\frac{dW}{dz} = u - iv = qe^{-i\theta}$. The Froude number F^2 has been defined in (2.15). The variables z, q and W have been normalized in (2.21). In these dimensionless variables, $q'_1 = 1$, $y'_1 = 1$ and $\psi'_1 = 1$. The dimensionless physical variable $z' = x' + iy'$ was given in (2.26) in terms of two functions W' and ω. That is

$$z' = \int e^{-\omega}\, dW' \qquad (7.1)$$

To express W' as a function of a single variable t, we use a conformal mapping (2.27). This should map the fluid region in the W'-plane (see Fig. 7.2) into the upper half-plane, the t-plane (see Fig. 7.3) with the boundary of the fluid region onto the real-axis, the boundary of the t-plane. To ensure the uniqueness of the mapping, we choose three corresponding points on each plane in the following way,

$$\begin{cases} A_\infty: & t = -\infty,\ W' = \infty\ , \\ B\ : & t = -1,\ W' = i\ , \\ D_\infty: & t = 0-,\ W' = -\infty. \end{cases} \tag{7.2}$$

The mapping is

$$W' = i + \frac{1}{\pi} \log\ (-t)\ , \tag{7.3}$$

where the constants are chosen so that for any real $t \geq 0$, $\mathrm{Im}(W') = 0$. (Note that the free surface $D_\infty\ B\ A_\infty$ corresponds to $\psi' = \psi'_1 = 1$.)

To express ω as a function of the single variable t, we make use of the Riemann-Hilbert method for a mixed-boundary-value problem in the upper half-plane. The general solution of the Riemann-Hilbert problem for an analytic function Q(t) in the upper half-plane was given in (2.28). Note that Q(t) in (2.28) is expressed in terms of the imaginary part of Q along the real-axis, the boundary of the t-plane. Now, we examine $\omega(t)$ along the real-axis of the t-plane, and we find that

$$\begin{cases} \text{Re } [\omega(t)] = \frac{1}{2} \log [1 - \frac{2}{F^2} (y'(t)-1)], & -\infty<t<0, \\ \text{Im } [\omega(t)] = 0 , & 0<t<\infty . \end{cases} \tag{7.4}$$

Note that we know either the imaginary or real part of $\omega(t)$ along the real-axis of the t-plane. Thus we have to construct an auxiliary function $H(t)$ which makes the imaginary part of the quotient $Q(t) = \frac{\omega(t)}{H(t)}$ known on the entire real-axis. One such $H(t)$ is

$$H(t) = \sqrt{t} . \tag{7.5}$$

Using (7.4), (7.5) and $Q(t) = \frac{\omega(t)}{H(t)}$, we obtain

$$\text{Im}[Q(t)] = \begin{cases} -\frac{1}{2\sqrt{-t}} \log[1-\frac{2}{F^2} (v'(t)-1)], & -\infty<t<0 \\ 0 , & 0<t<\infty \end{cases} \tag{7.6}$$

The final form of $\omega(t)$, $-\infty<t<0$, is

$$\omega(t) = \frac{-H(t)}{2\pi} \int_{-\infty}^{0} \frac{\log[1-\frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\ \sqrt{-\zeta}}$$

$$= \frac{-i\sqrt{-t}}{2\pi} \int_{-\infty}^{0} \frac{\log[1-\frac{2}{F^2} (y'(\zeta)-1)]d\zeta}{(\zeta-t)\ \sqrt{-\zeta}} \tag{7.7}$$

Note that (7.7) is obtained from (2.28), so that the series drops out when we apply the upstream condition, that is $|t|\to\infty$.

7.2 Solution of the Problem

The function, $\omega(t)$ along the free surface $D_\infty B\ A_\infty$, that is $-\infty<t<0$, is given in (7.7). Now, we restrict t to the interval (-1,0), that is, to the surface $D_\infty B$. Equation (7.7) can be written the following form

$$\omega(t) = \frac{-\sqrt{-t}}{2\pi}i\int_{-\infty}^{0} \frac{\log[1-\frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\sqrt{-\zeta}} + \frac{1}{2}\log[1-\frac{2}{F^2}(y'(t)-1)], \tag{7.8}$$

where $\int_{-\infty}^{0} = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{0}\right]$ with $\varepsilon>0$.

Equating real and imaginary parts of (7.8) and (2.23), we obtain

$$\begin{cases} \theta(t) = \dfrac{\sqrt{-t}}{2\pi}\displaystyle\int_{-\infty}^{0} \frac{\log.[1-\frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\sqrt{-\zeta}}, \\ q(t) = [1-\frac{2}{F^2}(y'(t)-1)]^{\frac{1}{2}} . \end{cases} \tag{7.9}$$

The dimensionless variable $z' = x' + iy'$ can be computed using (7.1), (7.3) and (7.9),

$$z' = x' + iy' = z'_0 + \int_{z'_o}^{z'} dz'$$

$$= z'_o + \frac{1}{\pi}\int_{-1}^{t} \frac{\exp[i\theta(\zeta)d\zeta}{\zeta q'(\zeta)}$$

$$= z_0' + \frac{1}{\pi}\int_{-1}^{t} \frac{\cos[\theta(\zeta)] + i\sin[\theta(\zeta)]\, d\zeta}{\zeta q'(\zeta)}, \tag{7.10}$$

and hence we obtain,

$$\begin{cases} x' = \dfrac{1}{\pi}\displaystyle\int_{-1}^{t} \dfrac{\cos[\theta(\zeta)]\, d\zeta}{\zeta q'(\zeta)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (-1<t\leq 0) \\ y' = \dfrac{1}{\pi}\displaystyle\int_{-1}^{t} \dfrac{\sin[\theta(\zeta)]d\zeta}{\zeta q'(\zeta)} + 1 + a', \end{cases} \tag{7.11}$$

In particular, the amplitude of the wave is given by

$$a' = -\frac{1}{\pi}\int_{-1}^{0} \frac{\sin[\theta(\zeta)]d\zeta}{\zeta q'(\zeta)} \tag{7.12}$$

7.3 Numerical Solution

The flow is symmetric with respect to the vertical line BC. Suppose we have two points $W'_1 = i + \frac{1}{\pi} \log(-t_1)$ and $W'_2 = i + \frac{1}{\pi} \log(-t_2)$ on the free surface (see Fig. 7.4), with $\mathrm{Re}(W'_1) = -\mathrm{Re}(W'_2)$. Then we have

$$t_1 = \frac{1}{t_2} \quad (7.13)$$

That is, W'_1 and W'_2 are inverse points in the t-plane.

Thus,

$$\begin{cases} \alpha'(t) = \alpha'(\frac{1}{t}), \\ \theta(t) = -\theta(\frac{1}{t}), \\ x(t) = -x(\frac{1}{t}), \\ y(t) = y(\frac{1}{t}), \end{cases} \qquad \text{for } -1<t<0 \quad (7.14)$$

Hence we only have to compute these variables on the free surface $D_\infty B$ or BA_∞. This overcomes the difficulty of integrating over the infinite range in the integral in (7.9).

The integral in (7.7) can be further simplified in the following way:

$$\int_{-\infty}^{0} \frac{\log[1 - \frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\sqrt{-\zeta}}$$

$$= \int_{-\infty}^{-1} \frac{\log[1-\frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\sqrt{-\zeta}} + \int_{-1}^{0} \frac{\log[1-\frac{2}{F^2}(y'(\zeta)-1)]d\zeta}{(\zeta-t)\sqrt{-\zeta}}$$

$$= \int_0^{-1} \frac{\log[1 - \frac{2}{F^2}(y'(\frac{1}{\zeta}) - 1)]}{(\frac{1}{\zeta} - t)\sqrt{-\frac{1}{\zeta}}} \left(\frac{-d\zeta}{\zeta^2}\right)$$

$$+ \int_{-1}^{0} \frac{\log[1 - \frac{2}{F^2}(y'(\zeta) - 1)]d\zeta}{(\zeta - t)\sqrt{-\zeta}}$$

$$= \int_{-1}^{0} \frac{\log[1 - \frac{2}{F^2}(y'(\frac{1}{\zeta}) - 1)]}{\sqrt{-\zeta}} \left\{\frac{1}{\zeta - t} - \frac{1}{1 - \zeta t}\right\} d\zeta \qquad (7.15)$$

Using (7.7) and (7.15), it is easy to show that $q'(t) = q'(\frac{1}{t})$ and $\theta(t) = -\theta(\frac{1}{t})$ for any $|t| > 0$.

The integrands in both equations (7.9) and (7.11) are non-linear and depend on the unknown values of y' along the free surface D_∞ B A_∞, and hence cannot be solved explicitly. Hence numerical methods must be introduced.

In computing $\theta(t)$ in (7.9), we assume that all y' along the free surface D_∞ B A_∞ are known. But this is not the case. Therefore, when we choose a particular quadrature formula, we still cannot compute the solution numerically. Hence an iterative method should be applied to solve the problem.

Initial approximation is found by choosing $g = 0$, that is the non-gravity case. In this case, the Froude number $F^2 = \infty$. In (7.9), $\theta(t) = 0$ and $q'(t) = 1$. The dimensionless physical variables x' and y' in (7.11) become

$$\begin{cases} x' = \frac{1}{\pi} \log |t| , & -1 < t < 0, \\ y' = 1, \end{cases} \qquad (7.16)$$

Two types of singularities appear in the computing procedure. The first type is when the integrand of (7.9) involves a square-root singularity at $t = 0$; and the second when the integrand has a simple pole within the range of integration. The method of overcoming these difficulties was discussed in the previous chapter.

We choose a number $(N+1)$ of points on $(-1,-\delta)$, $\delta>0$. These points are distributed equally on $(-1,-\delta)$. We then evaluate x' and y' at these points. For any particular point t, x' and y' can be found immediately using (7.14) or Lagrangian cubic-interpolation formula (3.15). Furthermore we know the integrand of (7.9) at $(N-1)$ points, the values at all other point t within the range of interests can be computed using (7.14) or (3.15).

If $\varepsilon>0$ and $\delta>0$ are given, with the initial approximation $(g = 0)$, we may start the iterative procedure. We compute $q(t)$ and $\theta(t)$ using (7.9), these values are then used to evaluate x' and y' in (7.11). We repeat the procedure until the successive approximation differ by a prescribed small number 10^{-k}, where $k=4$ or 6.

7.4 Numerical Results and Discussions

We choose $\varepsilon = 0.00125$ and $\delta = 0.0005$. For convenience, all numerical results are shown in the x' y' plane with origin at E (see Fig. 7.5), and only the shape of free surface B A_∞ will be shown and plotted. Tables 7.1 to 7.3 show some such free surfaces, which are plotted in Figs. 7.6-7.8. Table 7.4 show the relationship between F^2, a', q_B' and N, which is plotted in figure 7.9.

We use an iterative method, the iterative process becomes stable after the first iteration, and in the second iteration it produces 3 significant decimal places. For 6 significant decimal places we need 4 iterations. The CPU time for each iteration was approximately 0.5 seconds in IBM 360/65 computer.

TABLE 4.1

$\frac{h}{\lambda}$	q_1	ϕ_1	$\frac{c^2}{gh}$	$\frac{a}{h}$	$\frac{a}{\lambda}$
1.14271	0.8 0.6 0.5	0.43791 0.43915 0.44	0.14103 0.14725 0.15221	0.03058 0.06535 0.08277	0.03495 0.07568 0.09458
0.71911	0.8 0.6 0.5	0.69623 0.69933 0.70150	0.22402 0.23386 0.24170	0.04858 0.10386 0.13157	0.03493 0.07467 0.09461
0.6	0.8 0.6 0.5	0.83465 0.83913 0.84223	0.26827 0.28007 0.28947	0.05817 0.12434 0.15754	0.03490 0.07460 0.09453
0.3	0.8 0.6 0.5	1.67173 1.68860 1.70019	0.51289 0.53551 0.55356	0.11109 0.23655 0.29895	0.03333 0.07096 0.08968
0.15	0.8 0.6 0.5	3.34841 3.39526 3.42430	0.79467 0.83938 0.87180	0.17030 0.35620 0.44512	0.02555 0.05343 0.06677
0.10632	0.8 0.7 0.6 0.55 0.5	4.72602 4.75467 4.79044 4.80856 4.82	0.89777 0.92760 0.96671 0.98814 1.009[illegible]2	0.18943 0.29086 0.39189 0.44067 0.48965	0.02014 0.03092 0.04167 0.04686 0.05204

ϕ_1	q_1	a	d	h_1	h	g	λ	c	c_1
0.43791	0.8	0.03359	0.01585	1.08263	1.09848	5.35830	0.96128	0.91110	0.91110
0.43915	0.6	0.07724	0.03367	1.14829	1.18197	4.14267	1.03436	0.84912	0.84912
0.44000	0.5	0.10023	0.04148	1.16944	1.21092	3.74158	1.05969	0.83043	0.83041
0.69623	0.8	0.05339	0.02520	1.07385	1.09906	3.37132	1.52836	0.91108	0.91108
0.69933	0.6	0.12302	0.05368	1.13106	1.18474	2.60130	1.64750	0.84896	0.84895
0.70150	0.5	0.15993	0.06634	1.14920	1.21554	2.34484	1.69034	0.83001	0.83000
0.83465	0.8	0.06395	0.03018	1.06915	1.09933	2.81461	1.83220	0.91109	0.91109
0.83913	0.6	0.14747	0.06433	1.12170	1.18603	2.17000	1.97673	0.84901	0.84901
0.84223	0.5	0.19182	0.07952	1.13804	1.21757	1.95500	2.02925	0.83009	0.83010
1.67173	0.8	0.12222	0.05722	1.04304	1.10026	1.47270	3.66759	0.91162	0.91185
1.68860	0.6	0.28157	0.12057	1.06973	1.19031	1.13650	3.96786	0.85114	0.85202
1.70019	0.5	0.36597	0.14811	1.07607	1.22418	1.02468	4.08062	0.83330	0.83455
3.34841	0.8	0.18667	0.08188	1.01422	1.09610	0.96429	7.30710	0.91648	0.91765
3.39526	0.6	0.41780	0.15603	1.01692	1.17294	0.76591	7.81978	0.86838	0.87235
3.42430	0.5	0.53356	0.18247	1.01621	1.19867	0.70283	7.99128	0.85701	0.86229
4.72602	0.8	0.20610	0.08239	1.00562	1.08801	0.87336	10.23364	0.92362	0.92525
4.75467	0.7	0.32608	0.11562	1.00546	1.12108	0.78202	10.54485	0.90180	0.90507
4.79044	0.6	0.44932	0.14174	1.00480	1.14653	0.71219	10.78367	0.88846	0.89337
4.80856	0.55	0.50958	0.15192	1.00445	1.15637	0.68439	10.87518	0.88432	0.88997
4.82000	0.5	0.57074	0.16270	1.00289	1.16559	0.65705	10.96620	0.87906	0.88668

TABLE 4.2

TABLE 4.3

i	q(i)	θ(i) (degrees)	x(i)	y(i)
1	1.00000000	-0.00000000	0.00000000	0.00000000
2	0.99953580	0.81871140	0.02997150	0.00021385
3	0.99814470	1.62781136	0.05996480	0.00085417
4	0.99582280	2.43966464	0.09000182	0.00192094
5	0.99255500	3.24859066	0.12010900	0.00341810
6	0.98833200	4.05569104	0.15030300	0.00534556
7	0.98313000	4.85686890	0.18061100	0.00770859
8	0.97692820	5.65442037	0.21105930	0.01050945
9	0.96969200	6.44248699	0.24167560	0.01375511
10	0.96139380	7.22404520	0.27249060	0.01744737
11	0.95198300	7.99092548	0.30353710	0.02159631
12	0.94142430	8.74661032	0.33485310	0.02620276
13	0.92964950	9.47965658	0.36647880	0.03127909
14	0.91661460	10.19342531	0.39846270	0.03682423
15	0.90223170	10.87192720	0.43085650	0.04285192
16	0.88644890	11.51694477	0.46372500	0.04935662
17	0.86915960	12.10580057	0.49713710	0.05635040
18	0.85031030	12.63601267	0.53118190	0.06381828
19	0.82978590	13.07435418	0.56595650	0.07176363
20	0.80755650	13.40623945	0.60158540	0.08015007
21	0.78354500	13.58242797	0.63820830	0.08895296
22	0.75783000	13.55874512	0.67600350	0.09808576
23	0.73053760	13.26347210	0.71516860	0.10744540
24	0.70211560	12.58638846	0.75594210	0.11682760
25	0.67342720	11.41934774	0.79856340	0.12592020
26	0.64600020	9.60817342	0.84323700	0.13425850
27	0.62246370	7.00052012	0.89002280	0.14113750
28	0.60619900	3.79540286	0.93864460	0.14574210
29	0.59999940	-0.00000000	0.98836690	0.14746510

$q_1=0.025$
$q_2=1.11111$

$\emptyset_1$	h_1	h_2	b	q_{0+h}	g	F^2
3.8	29.19	0.86	1.45	0.047	0.721	2.774
4.1	37.46	0.94	1.47	0.039	0.660	3.030
4.4	48.66	1.02	1.50	0.031	0.608	3.290

TABLE 5.1

$q_2=1.11111$

q_1	x_1	$\emptyset_1$	h_1	h_2	b	$q_{\theta+h}$	g	F^2
0.025	40.	4.4	48.66	1.017	1.50	0.031	0.608	3.289
0.0125	80.	4.5	67.074	1.044	1.502	0.02	0.592	3.378
0.01	100.	4.6	80.61	1.07	1.51	0.016	0.577	3.466
0.008	125.	4.65	91.56	1.085	1.512	0.013	0.570	3.509

TABLE 5.2

$q_2=1.15$

q_1	x_1	$\emptyset_1$	h_1	h_2	b	q_{0+h}	g	F^2
0.025	40.	4.54	49.81	1.044	1.4787	0.032	0.6172	3.240
0.0125	80.	4.65	65.00	1.060	1.4827	0.022	0.608	3.289
0.01	100.	4.7	77.78	1.089	1.4897	0.018	0.591	3.384
0.008	125.	4.75	86.364	1.100	1.4923	0.015	0.584	3.415

TABLE 5.3

C1= 0.1250

C2= 1.1111

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0642673	-0.0086088
3	0.1226838	-0.0340686
4	0.1760532	-0.0671017
5	0.2249720	-0.1045199
6	0.2693685	-0.1452936
7	0.3099763	-0.1880990
8	0.3469809	-0.2324259
9	0.3806213	-0.2778599
10	0.4111412	-0.3240491
11	0.4388774	-0.3706864
12	0.4639027	-0.4176282
13	0.4865865	-0.4646602
14	0.5067943	-0.5117481
15	0.5251594	-0.5586807
16	0.5407896	-0.6055832
17	0.5561662	-0.6520392
18	0.5658148	-0.6986721
19	0.5785053	-0.7443408
20	0.5839462	-0.7901570
21	0.5931368	-0.8350425
22	0.5956144	-0.8797234
23	0.5956145	-0.9237756
24	0.5956145	-0.9666446
25	0.5956146	-1.0176820

TABLE 5.4

C1= 0.0125

C2= 1.1111

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0657159	-0.0089121
3	0.1253355	-0.0352097
4	0.1797169	-0.0692898
5	0.2294793	-0.1078777
6	0.2743640	-0.1498921
7	0.3157359	-0.1939730
8	0.3531949	-0.2395922
9	0.3871912	-0.2863223
10	0.4179388	-0.3337993
11	0.4459296	-0.3817134
12	0.4711055	-0.4299120
13	0.4938814	-0.4781842
14	0.5141506	-0.5264859
15	0.5326215	-0.5746168
16	0.5481514	-0.6226850
17	0.5634542	-0.6703009
18	0.5730861	-0.7180443
19	0.5856309	-0.7648300
20	0.5910127	-0.8117098
21	0.6000170	-0.8576671
22	0.6024419	-0.9033725
23	0.6024420	-0.9484206
24	0.6024420	-0.9922610
25	0.6024421	-1.0444570

TABLE 5.5

C1= 0.0101
C2= 1.1111

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0671631	-0.0092268
3	0.1279705	-0.0363894
4	0.1833417	-0.0715514
5	0.2339212	-0.1113322
6	0.2796659	-0.1546086
7	0.3213748	-0.1999817
8	0.3592620	-0.2469055
9	0.3935915	-0.2949404
10	0.4246481	-0.3437105
11	0.4527748	-0.3929035
12	0.4780872	-0.4423595
13	0.5009456	-0.4918703
14	0.5212680	-0.5413837
15	0.5396419	-0.5907088
16	0.5552689	-0.6399395
17	0.5705027	-0.6887096
18	0.5801173	-0.7375615
19	0.5925277	-0.7854578
20	0.5978532	-0.8333991
21	0.6066865	-0.8804227
22	0.6090630	-0.9271513
23	0.6090631	-0.9731954
24	0.6090631	-1.0180060
25	0.6090632	-1.0713620

TABLE 5.6

C1= 0.0080

C2= 1.1111

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0678864	-0.0093880
3	0.1292853	-0.0369844
4	0.1881485	-0.0726891
5	0.2381324	-0.1130702
6	0.2822027	-0.1569805
7	0.3241749	-0.2030016
8	0.3622713	-0.2505795
9	0.3967622	-0.2992680
10	0.4279402	-0.3486857
11	0.4561579	-0.3985191
12	0.4815342	-0.4486043
13	0.5044299	-0.4987345
14	0.5247748	-0.5488539
15	0.5431470	-0.5987759
16	0.5587693	-0.6485876
17	0.5739660	-0.6979342
18	0.5835656	-0.7473400
19	0.5959109	-0.7957910
20	0.6012080	-0.8442628
21	0.6099533	-0.8918191
22	0.6123053	-0.9390591
23	0.6123055	-0.9856010
24	0.6123055	-1.0308980
25	0.6123055	-1.0848330

TABLE 5.7

C1= 0.0250

C2= 1.1500

I	X(I)	Y(I)
1	0.0000000	-0.0000000
2	0.0662016	-0.0090036
3	0.1264093	-0.0355062
4	0.1812171	-0.0698056
5	0.2313462	-0.1085923
6	0.2767507	-0.1507735
7	0.3192055	-0.1940915
8	0.3559181	-0.2407102
9	0.3901523	-0.2876039
10	0.4211733	-0.3350130
11	0.4493367	-0.3820362
12	0.4747308	-0.4311147
13	0.4977403	-0.4793303
14	0.5182422	-0.5275760
15	0.5368811	-0.5756139
16	0.5527796	-0.6235860
17	0.5684233	-0.6710714
18	0.5783044	-0.7187057
19	0.5913252	-0.7653462
20	0.5970048	-0.8121243
21	0.6065831	-0.8570316
22	0.6091615	-0.9035709
23	0.6091616	-0.9425809
24	0.6091616	-0.9924323
25	0.6091617	-1.0439230

TABLE 5.8

C1= 0.0125

C2= 1.1500

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0671604	-0.0091820
3	0.1279995	-0.0361862
4	0.1834133	-0.0711131
5	0.2340461	-0.1105993
6	0.2798597	-0.1535259
7	0.3216478	-0.1985043
8	0.3596268	-0.2449975
9	0.3940676	-0.2925689
10	0.4252479	-0.3406512
11	0.4535260	-0.3895330
12	0.4790028	-0.4384625
13	0.5020594	-0.4874279
14	0.5225902	-0.5363894
15	0.5412244	-0.5851443
16	0.5571160	-0.6336094
17	0.5727072	-0.6819865
18	0.5825744	-0.7302822
19	0.5954998	-0.7775885
20	0.6011413	-0.8243996
21	0.6106612	-0.8714536
22	0.6131461	-0.9176941
23	0.6131462	-0.9632988
24	0.6131462	-1.0077240
25	0.6131463	-1.0597940

TABLE 5.9

C1= 0.0100

C2= 1.1500

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0687513	-0.0095367
3	0.1308898	-0.0374994
4	0.1873819	-0.0736246
5	0.2389004	-0.1144204
6	0.2854251	-0.1587474
7	0.3277890	-0.2051484
8	0.3662244	-0.2530734
9	0.4010172	-0.3020795
10	0.4324670	-0.3517794
11	0.4609395	-0.4018605
12	0.4865561	-0.4521649
13	0.5096942	-0.5024838
14	0.5302767	-0.5527692
15	0.5489084	-0.6028265
16	0.5647933	-0.6527593
17	0.5803063	-0.7021048
18	0.5901539	-0.7516978
19	0.6029335	-0.8002194
20	0.6085162	-0.8487830
21	0.6177925	-0.8963938
22	0.6202857	-0.9437547
23	0.6202858	-0.9904411
24	0.6202858	-1.0359280
25	0.6202859	-1.0892420

TABLE 5.10

C1= 0.0080

C2= 1.1500

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0693296	-0.0096636
3	0.1319398	-0.0379772
4	0.1888221	-0.0745363
5	0.24[illegible]6601	-0.1158245
6	0.2874405	-0.1606502
7	0.3300102	-0.2075700
8	0.3680077	-0.2560195
9	0.4035247	-0.3055444
10	0.4350586	-0.3557603
11	0.4636080	-0.4063521
12	0.4892717	-0.4571573
13	0.5124362	-0.5079686
14	0.5330340	-0.5587342
15	0.5516619	-0.6092663
16	0.5675414	-0.6596603
17	0.5830233	-0.7095531
18	0.5928618	-0.7594954
19	0.6055853	-0.8084529
20	0.6111459	-0.8574409
21	0.6203533	-0.9054802
22	0.6228271	-0.9532402
23	0.6228272	-1.0003200
24	0.6228272	-1.0461940
25	0.6228272	-1.0996600

TABLE 5.11

F2	QE	1.2500000	1.3999990	1.6000000
	CP	0.5625000	0.9600000	1.5600000
0.1000000	B	0.3580000	0.3230000	0.2780000
0.2500000	B	0.3970000	0.3500000	0.2970000
0.4000000	B	0.4150000	0.3630000	0.3050000
0.6000000	B	0.4300000	0.3730000	0.3120000
0.8000000	B	0.4400000	0.3800000	0.3160000
1.0000000	B	0.4470000	0.3850000	0.3200000
1.1999990	B	0.4530000	0.3900000	0.3230000
1.3999990	B	0.4570000	0.3930000	0.3260000

TABLE 6.1

F2	QE	1.2000	1.2500	1.4000	1.6000	1.8000
	CP	0.4400	0.5625	0.9600	1.5600	2.2400
0.5500000	B	0.4580	0.4270	0.3710	0.3100	0.2620
0.8000000	B	0.4620	0.4400	0.3800	0.3160	0.2670
1.1999990	B	0.4470	0.4530	0.3900	0.3230	0.2720

TABLE 6.2

QE= 1.2500 ε = 0.0250
CP= 0.5625 F2= 1.0000

δ	0.0050	0.0020	0.0010	0.0005	0.0001
B	0.4210	0.4290	0.4370	0.4470	0.4790
−Y	1.1670	1.8120	2.6730	4.0650	11.4440

TABLE 6.3

QE= 1.2500 δ = 0.0005
CP= 0.5625 F2= 1.0000

ε	0.1000	0.0750	0.0500	0.0250	0.0150	0.0100
B	0.5610	0.4950	0.4620	0.4470	0.4320	0.4140
−Y	4.0380	4.0570	4.0650	4.0650	4.0640	4.0630

TABLE 6.4

QE= 1.1999990

CP= 0.4339986

F2= 1.1999990

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0135868	-0.0004759
3	0.0277138	-0.0025251
4	0.0421209	-0.0072244
5	0.0570678	-0.0126984
6	0.0724937	-0.0199025
7	0.0885427	-0.0279453
8	0.1051846	-0.0378770
9	0.1225818	-0.0488899
10	0.1407372	-0.0621390
11	0.1598437	-0.0769355
12	0.1799574	-0.0946019
13	0.2013291	-0.1146734
14	0.2241121	-0.1388171
15	0.2486865	-0.1670924
16	0.2753936	-0.2020633
17	0.3049276	-0.2454040
18	0.3362347	-0.3030046
19	0.3770960	-0.3847704
20	0.4074417	-0.5294123
21	0.4767897	-4.3065950

TABLE 6. 5

QE= 1.3999990

CP= 0.5599981

F2= 1.1999999

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0116577	0.0002269
3	0.0238052	-0.0004197
4	0.0362079	-0.0045835
5	0.0490428	-0.0090875
6	0.0622937	-0.0155740
7	0.0760351	-0.0224555
8	0.0902828	-0.0314610
9	0.1051265	-0.0410790
10	0.1206107	-0.0531380
11	0.1368462	-0.0662372
12	0.1539214	-0.0823621
13	0.1719896	-0.1003181
14	0.1912150	-0.1224126
15	0.2118410	-0.1479360
16	0.2341846	-0.1800303
17	0.2586960	-0.2194852
18	0.2861218	-0.2725323
19	0.3175776	-0.3476245
20	0.3559148	-0.4808182
21	0.3896837	-4.1798310

TABLE 6. 6

OE= 1.6000000

CP= 1.5600000

F2= 1.1999990

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0101593	0.0006888
3	0.0207340	0.0009899
4	0.0316481	-0.0027207
5	0.0428800	-0.0064756
6	0.0545208	-0.0123325
7	0.0665192	-0.0182807
8	0.0789986	-0.0264562
9	0.0919213	-0.0349176
10	0.1054341	-0.0458948
11	0.1195158	-0.0575492
12	0.1343476	-0.0722554
13	0.1499410	-0.0883664
14	0.1665351	-0.1085536
15	0.1842070	-0.1316185
16	0.2033053	-0.1610038
17	0.2240412	-0.1968986
18	0.2470611	-0.2456012
19	0.2729033	-0.3144017
20	0.2820781	-0.4363313
21	0.3233531	-4.0474970

TABLE 6. 7

OE= 1.8000000

CP= 2.2399990

F2= 1.1999990

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0089629	0.0009977
3	0.0182607	0.0019529
4	0.0280258	-0.0013784
5	0.0379998	-0.0045482
6	0.0484022	-0.0098653
7	0.0590398	-0.0150616
8	0.0701635	-0.0225162
9	0.0815934	-0.0300252
10	0.0935994	-0.0400546
11	0.1060148	-0.0504966
12	0.1191369	-0.0639510
13	0.1328241	-0.0784869
14	0.1474177	-0.0969774
15	0.1628229	-0.1179069
16	0.1794577	-0.1448598
17	0.1973072	-0.1776039
18	0.2169836	-0.2223586
19	0.2385402	-0.2854587
20	0.2408316	-0.3968385
21	0.2722387	-3.9122420

TABLE 6. 8

OE= 1.1999990

CP= 0.4399986

F2= 1.0000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0135880	-0.0004534
3	0.0277222	-0.0024410
4	0.0421492	-0.0070561
5	0.0571265	-0.0124180
6	0.0725803	-0.0194832
7	0.0886677	-0.0273547
8	0.1053584	-0.0370822
9	0.1228086	-0.0478494
10	0.1410286	-0.0608087
11	0.1602041	-0.0752591
12	0.1803994	-0.0925162
13	0.2018570	-0.1120949
14	0.2247496	-0.1356468
15	0.2494112	-0.1631933
16	0.2762463	-0.1972561
17	0.3058879	-0.2394196
18	0.3393090	-0.2954245
19	0.3782326	-0.3748031
20	0.4077853	-0.5151215
21	0.4703941	-4.0920969

TABLE 6. 9

OE= 1.3999990

CP= 0.5599981

F2= 1.0000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0116573	0.0002435
3	0.0238079	-0.0003686
4	0.0362255	-0.0044799
5	0.0490783	-0.0089136
6	0.0623559	-0.0153127
7	0.0761274	-0.0220861
8	0.0904156	-0.0309623
9	0.1053037	-0.0404237
10	0.1208448	-0.0522977
11	0.1371425	-0.0651738
12	0.1542952	-0.0810339
13	0.1724482	-0.0986689
14	0.1917782	-0.1203731
15	0.2125196	-0.1454111
16	0.2350049	-0.1768946
17	0.2596748	-0.2155447
18	0.2872949	-0.2674837
19	0.3189668	-0.3408782
20	0.3370611	-0.4728717
21	0.3854437	-3.9824040

TABLE 6.10

QE= 1.6000000
CP= 1.5600000
F2= 1.0000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0101585	0.0006974
3	0.0207341	0.0010223
4	0.0316590	-0.0026544
5	0.0429034	-0.0063638
6	0.0545639	-0.0121642
7	0.0665843	-0.0180420
8	0.0790943	-0.0261334
9	0.0920509	-0.0344927
10	0.1056080	-0.0453489
11	0.1197388	-0.0568569
12	0.1346335	-0.0713891
13	0.1502970	-0.0872869
14	0.1669798	-0.1072153
15	0.1847522	-0.1299550
16	0.2039782	-0.1589286
17	0.2248646	-0.1942747
18	0.2480788	-0.2422133
19	0.2741654	-0.3098220
20	0.2833761	-0.4294349
21	0.3262095	-3.8665800

TABLE 6.11

QE= 1.8000000
CP= 2.2399990
F2= 1.0000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0089620	0.0010032
3	0.0182599	0.0019731
4	0.0280326	-0.0013347
5	0.0380155	-0.0044742
6	0.0484320	-0.0097534
7	0.0590855	-0.0149028
8	0.0702317	-0.0223911
9	0.0816866	-0.0297416
10	0.0937259	-0.0396900
11	0.1061782	-0.0500336
12	0.1193485	-0.0633707
13	0.1330898	-0.0777627
14	0.1477532	-0.0960780
15	0.1632389	-0.1167862
16	0.1799781	-0.1444579
17	0.1979534	-0.1758241
18	0.2177978	-0.2200485
19	0.2395771	-0.2823105
20	0.2420226	-0.3920255
21	0.2696661	-3.7475790

TABLE 6.12

OE= 1.1999990

CP= 0.4399986

F2= 0.8000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0135896	-0.0004225
3	0.0277332	-0.0023260
4	0.0421863	-0.0068255
5	0.0571889	-0.0120331
6	0.0726920	-0.0189073
7	0.0888268	-0.0265432
8	0.1055775	-0.0359903
9	0.1230909	-0.0464208
10	0.1413872	-0.0589843
11	0.1606407	-0.0729625
12	0.1809277	-0.0896636
13	0.2024769	-0.1085753
14	0.2254639	-0.1313296
15	0.2502356	-0.1578988
16	0.2771726	-0.1907514
17	0.3069103	-0.2313562
18	0.3404037	-0.2852653
19	0.3793146	-0.3615386
20	0.4077498	-0.4962844
21	0.4622688	-3.8384880

TABLE 6.13

OE= 1.3999990

CP= 0.5599981

F2= 0.8000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0116568	0.0002594
3	0.0238115	-0.0002976
4	0.0362496	-0.0043357
5	0.0491265	-0.0086711
6	0.0624399	-0.0149482
7	0.0762509	-0.0215701
8	0.0905923	-0.0302654
9	0.1055377	-0.0395081
10	0.1211521	-0.0511239
11	0.1375278	-0.0636893
12	0.1547779	-0.0791813
13	0.1730347	-0.0963688
14	0.1924920	-0.1175358
15	0.2133699	-0.1419053
16	0.2360250	-0.1725517
17	0.2608690	-0.2101047
18	0.2887000	-0.2605442
19	0.3205919	-0.3316627
20	0.3383013	-0.4574119
21	0.3801869	-3.7477510

TABLE 6.14

QE= 1.6000000

CP= 1.5600000

F2= 0.8000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0101572	0.0007094
3	0.0207344	0.0010677
4	0.0316743	-0.0025610
5	0.0429363	-0.0062059
6	0.0546238	-0.0119259
7	0.0666744	-0.0177040
8	0.0792261	-0.0256759
9	0.0922281	-0.0338903
10	0.1058451	-0.0445749
11	0.1200408	-0.0558755
12	0.1350186	-0.0731611
13	0.1507733	-0.0857580
14	0.1675715	-0.1053219
15	0.1854727	-0.1276042
16	0.2048613	-0.1560011
17	0.2259346	-0.1905819
18	0.2493870	-0.2374613
19	0.2757648	-0.3034305
20	0.2849815	-0.4198910
21	0.3164553	-3.6532410

TABLE 6.15

QE= 1.8000000

CP= 2.2399999

F2= 0.8000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0089606	0.0010110
3	0.0182588	0.0020033
4	0.0280424	-0.0012724
5	0.0380378	-0.0043684
6	0.0484743	-0.0095933
7	0.0591500	-0.0146751
8	0.0703278	-0.0219925
9	0.0818170	-0.0293348
10	0.0939025	-0.0391666
11	0.1064055	-0.0493691
12	0.1196418	-0.0625380
13	0.1334565	-0.0767238
14	0.1482141	-0.0947886
15	0.1638076	-0.1151809
16	0.1876856	-0.1414518
17	0.1988265	-0.1732820
18	0.2188899	-0.2167575
19	0.2409536	-0.2778435
20	0.2435869	-0.3852451
21	0.2666767	-3.5487669

TABLE 6.16

OE= 1.1999990

CP= 0.4399986

F2= 0.6000003

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0135917	-0.0003775
3	0.0277482	-0.0021577
4	0.0422377	-0.0064868
5	0.0572819	-0.0114669
6	0.0728415	-0.0180597
7	0.0890355	-0.0253489
8	0.1058605	-0.0343844
9	0.1234463	-0.0443219
10	0.1418301	-0.0563081
11	0.1611658	-0.0696006
12	0.1815477	-0.0854984
13	0.2031807	-0.1034514
14	0.2262596	-0.1250671
15	0.2511014	-0.1502504
16	0.2781027	-0.1814007
17	0.3078551	-0.2198329
18	0.3413091	-0.2708490
19	0.3800146	-0.3428892
20	0.4068061	-0.4701145
21	0.4513095	-3.5292920

TABLE 6.17

OE= 1.3999990

CP= 0.5599981

F2= 0.6000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0116559	0.0002874
3	0.0238168	-0.0001919
4	0.0362850	-0.0041199
5	0.0491962	-0.0083072
6	0.0625603	-0.0144001
7	0.0764256	-0.0207937
8	0.0908400	-0.0292166
9	0.1058610	-0.0381301
10	0.1215723	-0.0493581
11	0.1380473	-0.0614582
12	0.1554209	-0.0764010
13	0.1738033	-0.0929247
14	0.1934141	-0.1132960
15	0.2144472	-0.1366819
16	0.2372841	-0.1661056
17	0.2623163	-0.2020683
18	0.2903554	-0.2533556
19	0.3224221	-0.3182458
20	0.3394799	-0.4380546
21	0.3732291	-3.4583610

TABLE 6.18

QE= 1.6000000
CP= 1.5600000
F2= 0.6000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0101554	0.0007275
3	0.0207350	0.0011366
4	0.0316977	-0.0024187
5	0.0429856	-0.0059646
6	0.0547134	-0.0115611
7	0.0668074	-0.0171857
8	0.0794197	-0.0249739
9	0.0924858	-0.0329655
10	0.1061876	-0.0433868
11	0.1204731	-0.0543693
12	0.1355659	-0.0682779
13	0.1514431	-0.0834154
14	0.1683959	-0.1024249
15	0.1864644	-0.1240155
16	0.2060630	-0.1515440
17	0.2273690	-0.1849798
18	0.2511145	-0.2302878
19	0.2778256	-0.2938520
20	0.2869679	-0.4057518
21	0.3116932	-3.38[illegible]4430

TABLE 6.19

QE= 1.6000000
CP= 2.2399990
F2= 0.6000000

I	X(I)	Y(I)
1	0.0000000	0.0000000
2	0.0089586	0.0010231
3	0.0182572	0.0020492
4	0.0280578	-0.0011759
5	0.0380723	-0.0042341
6	0.0485393	-0.0093441
7	0.0592483	-0.0143206
8	0.0704733	-0.0215116
9	0.0820133	-0.0287002
10	0.0941679	-0.0383502
11	0.1067436	-0.0483322
12	0.1200756	-0.0612391
13	0.1339948	-0.0751039
14	0.1488869	-0.0927891
15	0.1646306	-0.1126835
16	0.1817017	-0.1383374
17	0.2000669	-0.1693456
18	0.2204244	-0.2116812
19	0.2428575	-0.2709938
20	0.2457108	-0.3749532
21	0.2630577	-3.2982699

TABLE 6.20

k	$x'(t_k)$	$y'(t_k)-1$
0	0.000000	0.797057
20	0.101991	0.772497
40	0.180239	0.736234
50	0.214270	0.719156
100	0.365041	0.642934
150	0.505749	0.575489
200	0.649174	0.511980
250	0.803244	0.449844
300	0.976588	0.387237
350	1.182249	0.322368
400	1.445343	0.252888
450	1.830670	0.174547
500	2.661923	0.074300
520	4.642400	0.006592

Table 7.1 Shape of the solitary wave when $q'_B = 0.20$ and $N = 520$.

k	$x'(t_k)$	$y'(t_k)-1$
0	0.000000	0.790109
20	0.085331	0.775144
40	0.157261	0.746900
60	0.218213	0.718415
80	0.273515	0.691315
100	0.325663	0.665540
120	0.376005	0.640844
140	0.425378	0.617002
160	0.474365	0.593825
180	0.523413	0.571164
200	0.572892	0.548894

Table 7.2 Shape of the solitary wave near crest when $q_B' = 0.222$ and $N = 600$ (the fastest wave).

k	$x'(t_k)$	$y'(t_k)-1$
0	0.000000	0.763683
20	0.071504	0.756945
40	0.141157	0.739298
60	0.203865	0.717111
80	0.261500	0.693621
100	0.315820	0.670002
120	0.368056	0.646638
140	0.419064	0.623625
160	0.469469	0.600959
180	0.519756	0.578593
200	0.570329	0.556468

Table 7.3 Shape of the solitary wave near crest when $q'_B = 0.28$ and $N = 600$.

N	$q_B'=0.226$ F^2	$q_B'=0.224$ F^2	$q_B'=0.222$ F^2	$q_B'=0.220$ F^2	$q_B'=0.218$ F^2
300	1.655028	1.655061	1.655085	1.655100	1.655107
450	1.659523	1.659548	1.659564	1.659572	1.659570
600	1.662101	1.662123	1.662136	1.662140	1.662134
	1.671166	1.671183	1.671191	1.671181	1.671165

N	$q_B'=0.226$ a'	$q_B'=0.224$ a'	$q_B'=0.222$ a'	$q_B'=0.220$ a'	$q_B'=0.218$ a'
300	0.785248	0.786008	0.786758	0.787497	0.788225
450	0.787380	0.788139	0.788887	0.789624	0.790350
600	0.788604	0.789362	0.790109	0.790849	0.791571
	0.792912	0.793665	0.794409	0.795174	0.795872

Table 7.4 Relationships between F^2, a', q_B' and N for the solitary waves

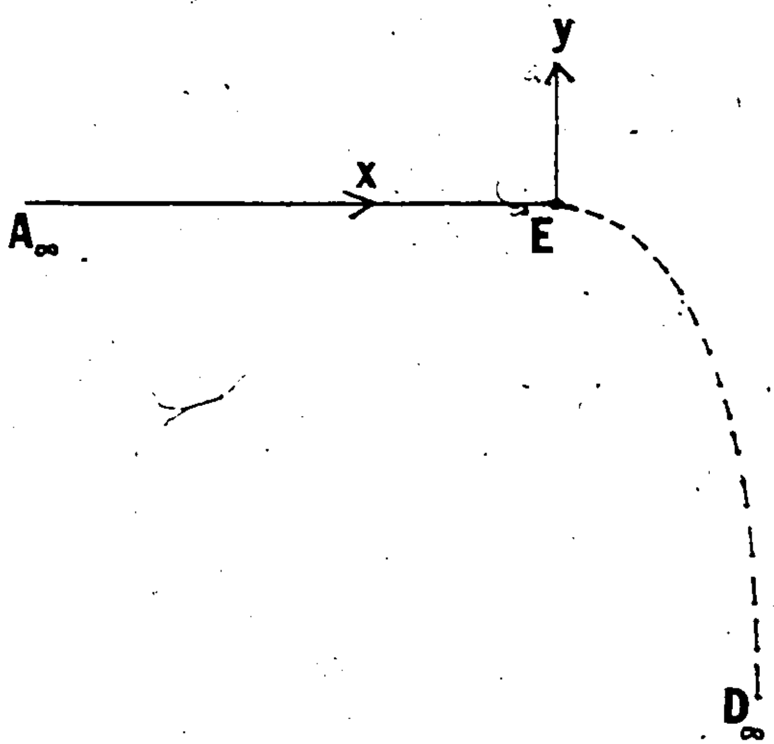


FIGURE 2.1

A streamline consisting of a horizontal line and a free surface

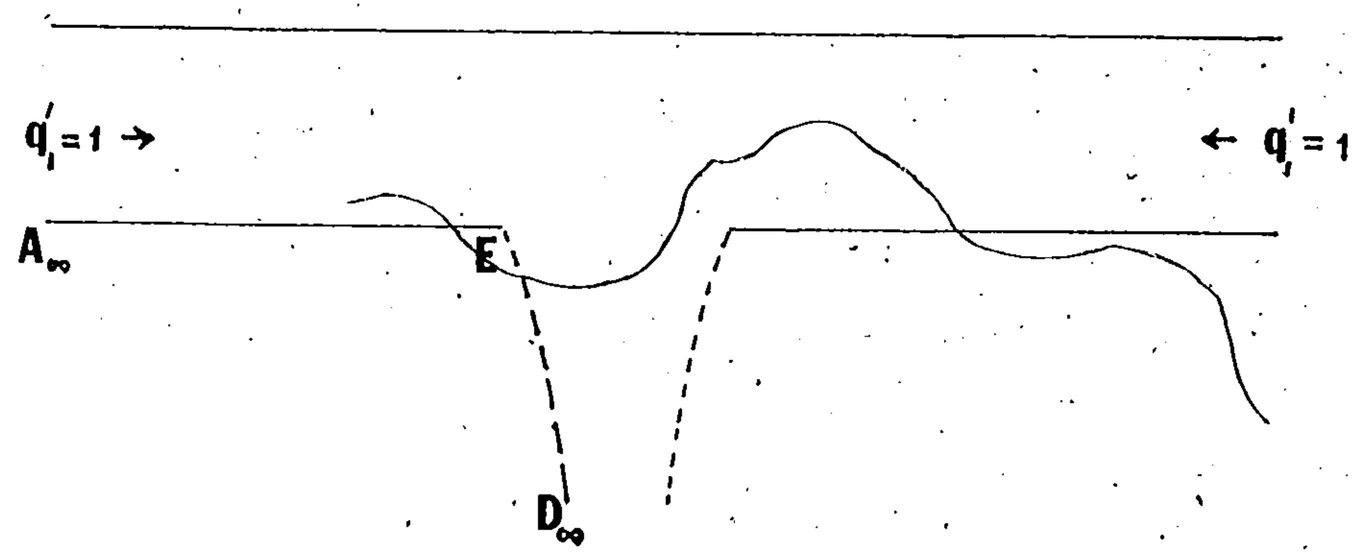


FIGURE 2. 2

A 2-D vertical jet from an infinite channel

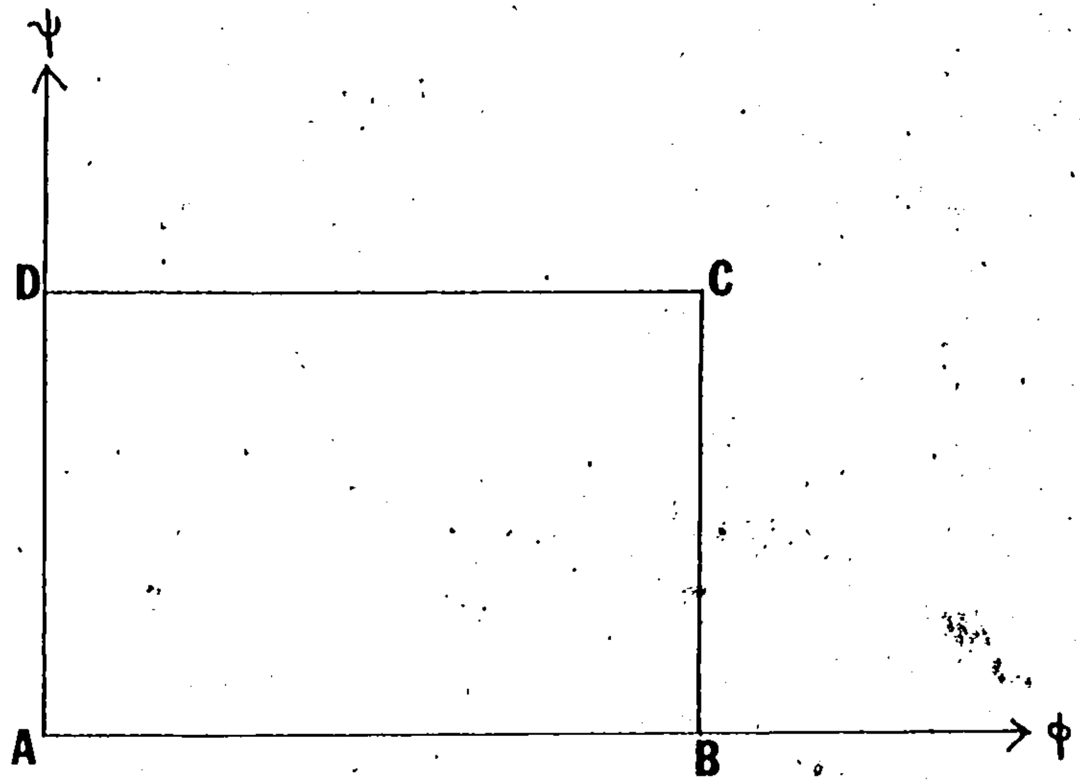


FIGURE 4. 1

A rectangular region in the W-plane

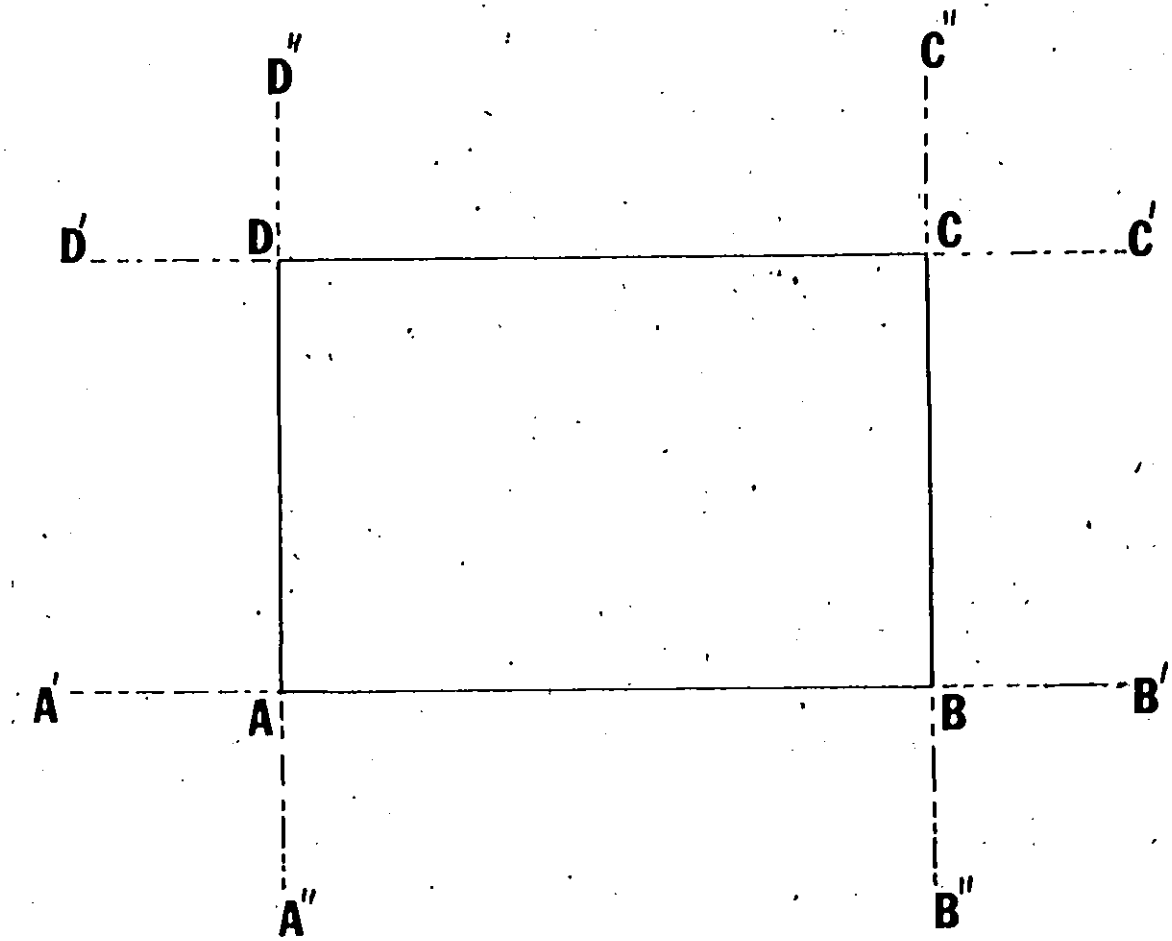


FIGURE 3. 3

A rectangular region and sides outside the rectangle

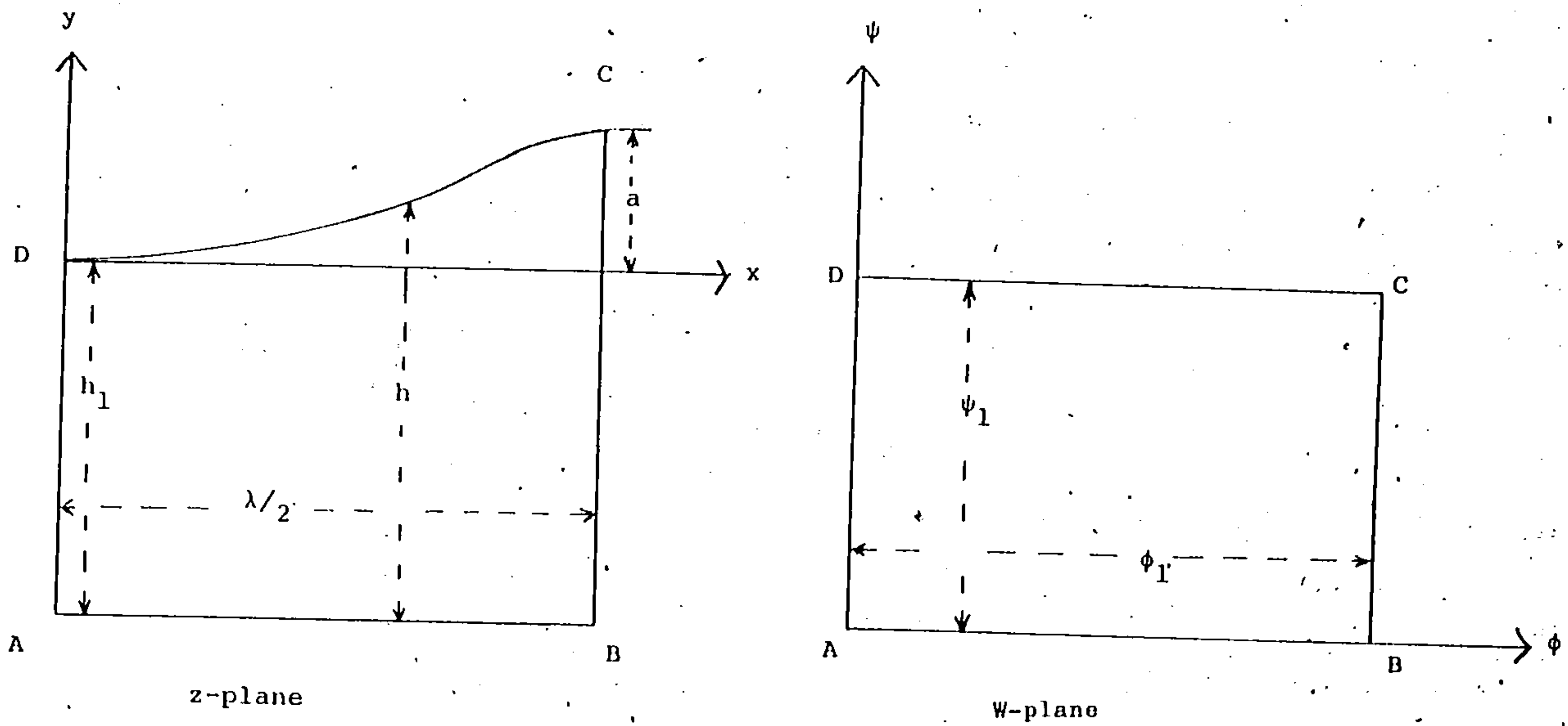
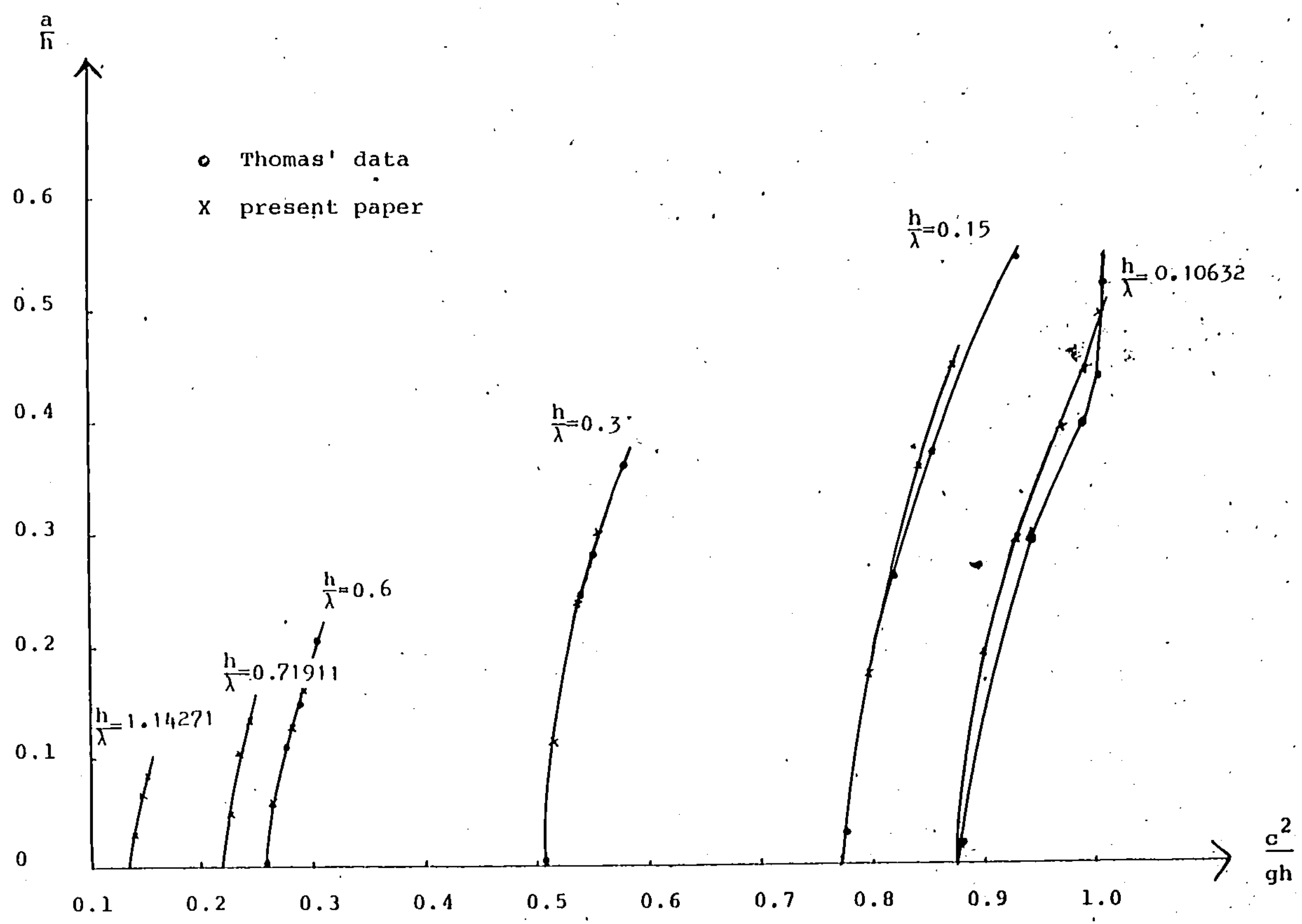


Figure 4.1



a/h
0.6
0.5
0.4
0.3
0.2
0.1
0
o Thomas' data
X present paper
h/λ=1.14271
h/λ=0.71911
h/λ=0.6
h/λ=0.3
h/λ=0.15
h/λ=0.10632
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
c²/gh

Figure 4.2

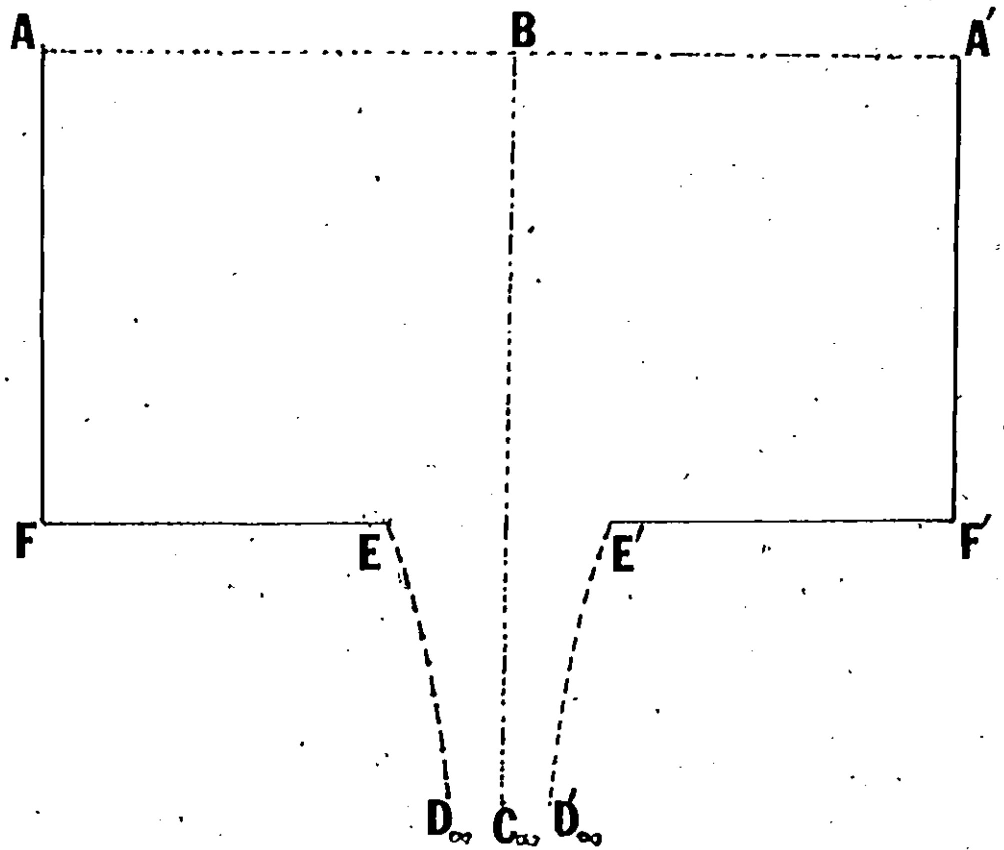


FIGURE [illegible]. 1a

The physical plane for a 2-D vertical jet from a vessel

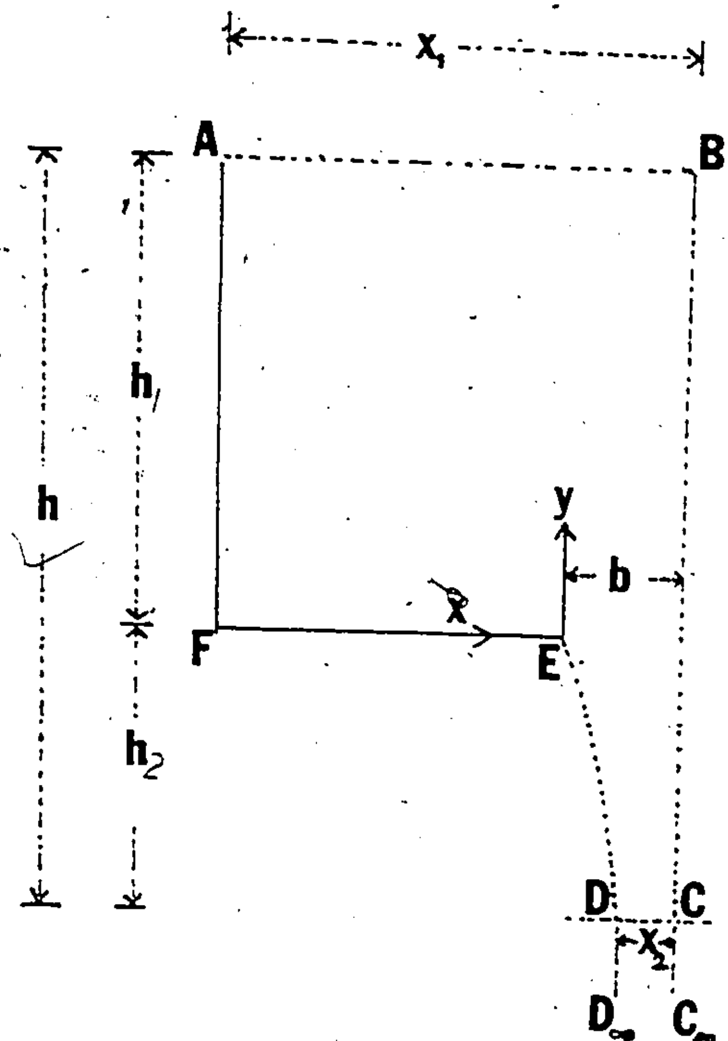


FIGURE 5. 16

Half of the physical plane for a 2-D vertical jet from a vessel

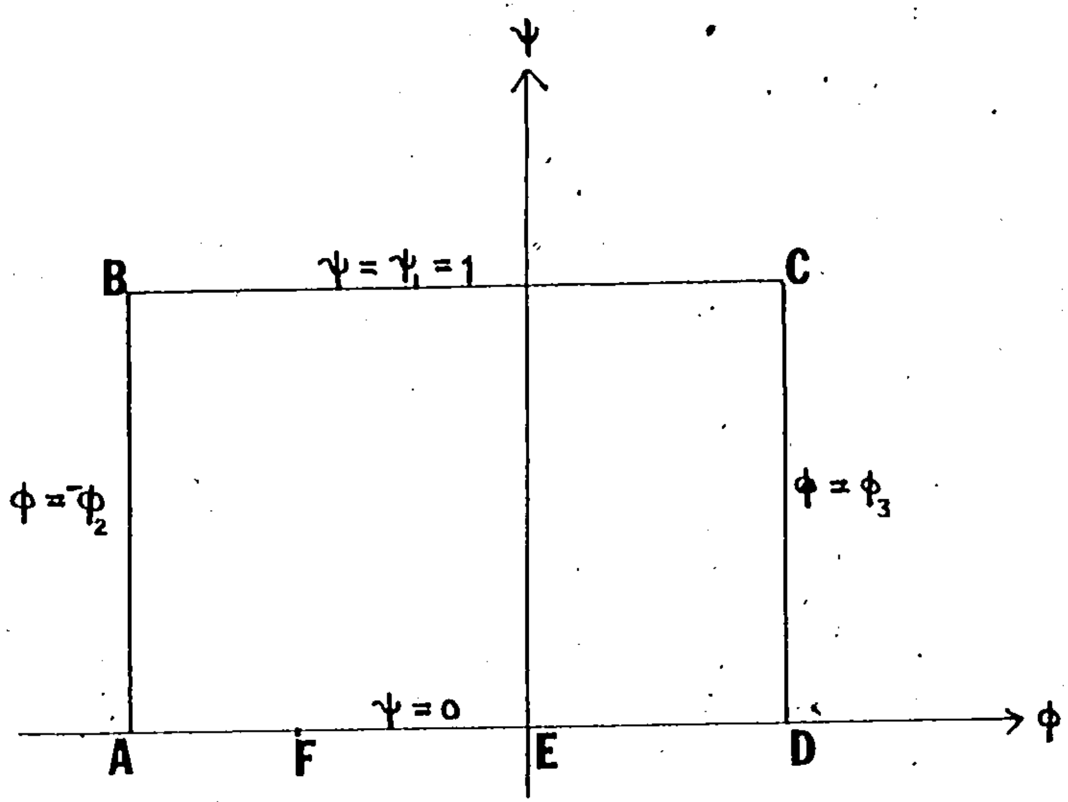


FIGURE 5. 2

The W-plane for a 2-D vertical jet from a vessel

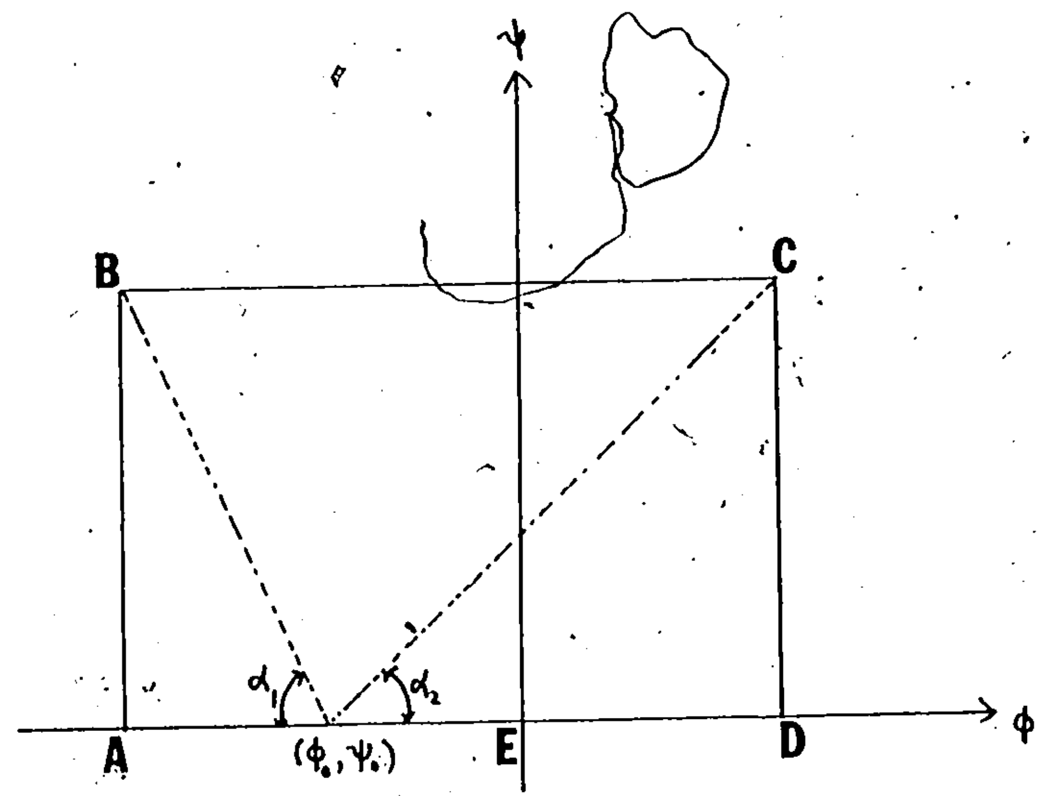


FIGURE 5. !

α_1 and α_2 used in formula (5.5)

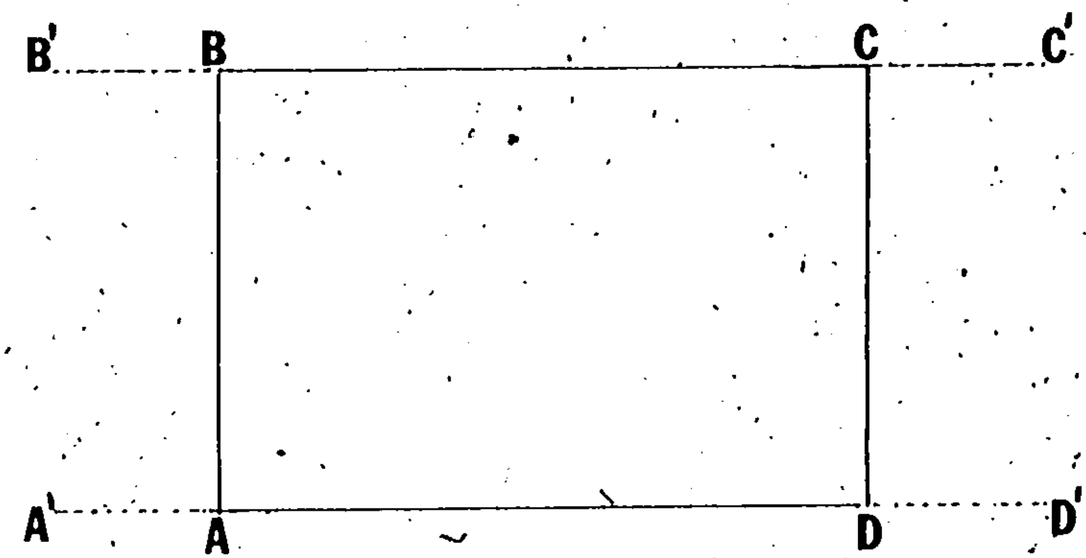


FIGURE 5.4

The reflections of V outside the rectangle ABCD

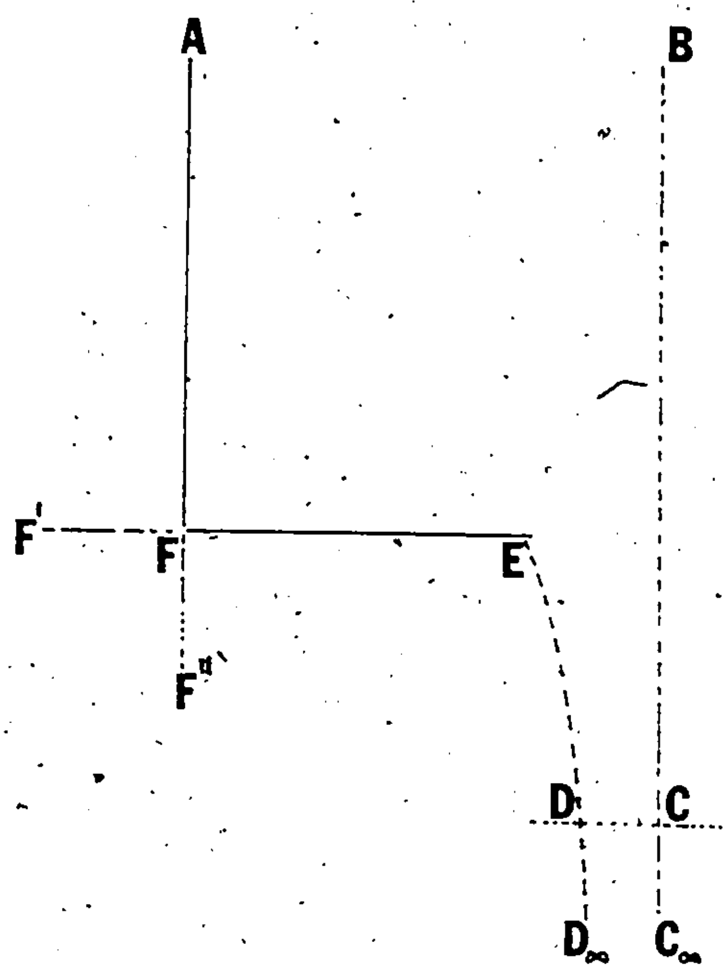


FIGURE 5. 5
The reflections of U or V about the point F

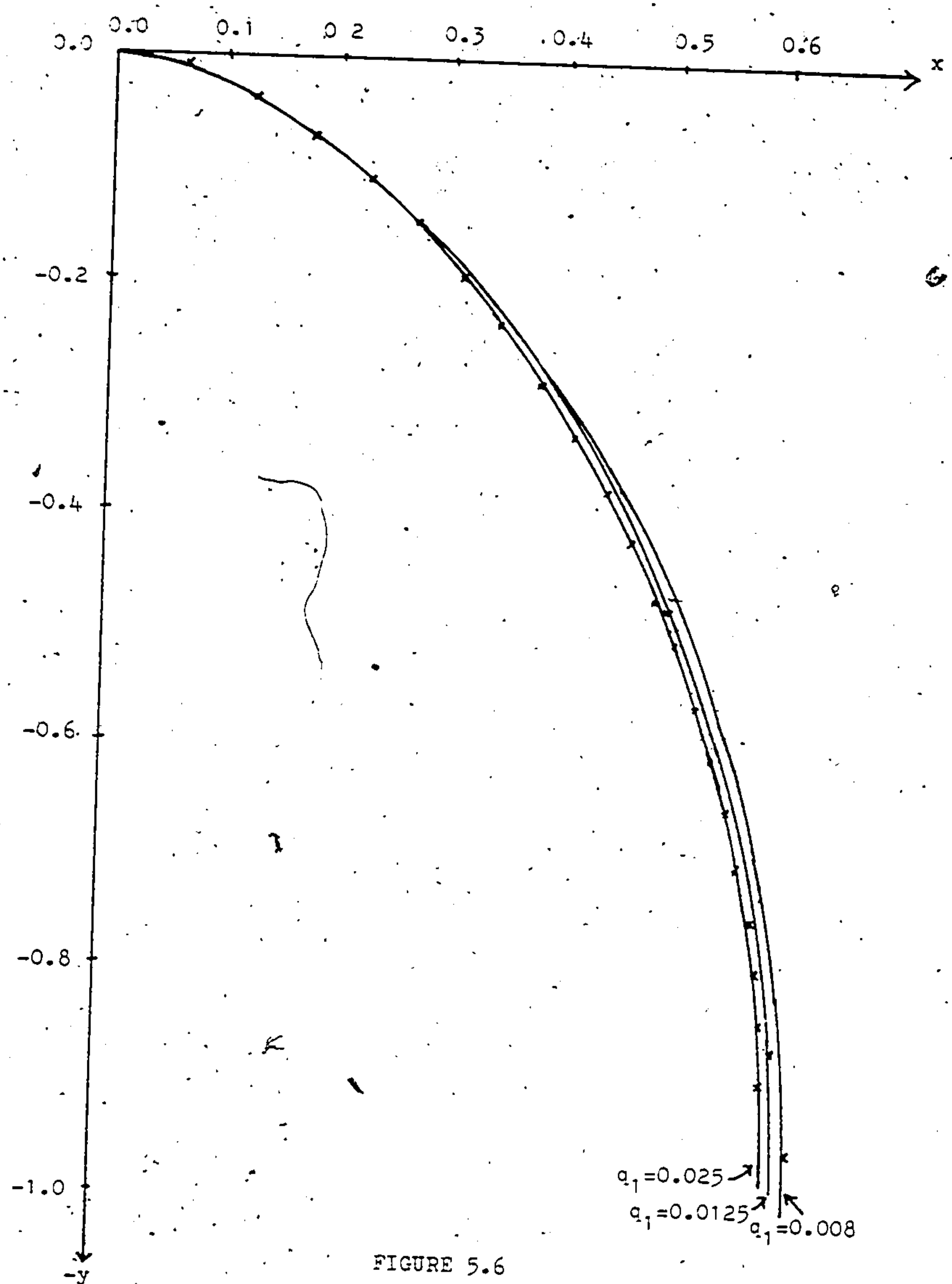


FIGURE 5.6

Shapes of jet near aperture E (when q_2=1.11111)

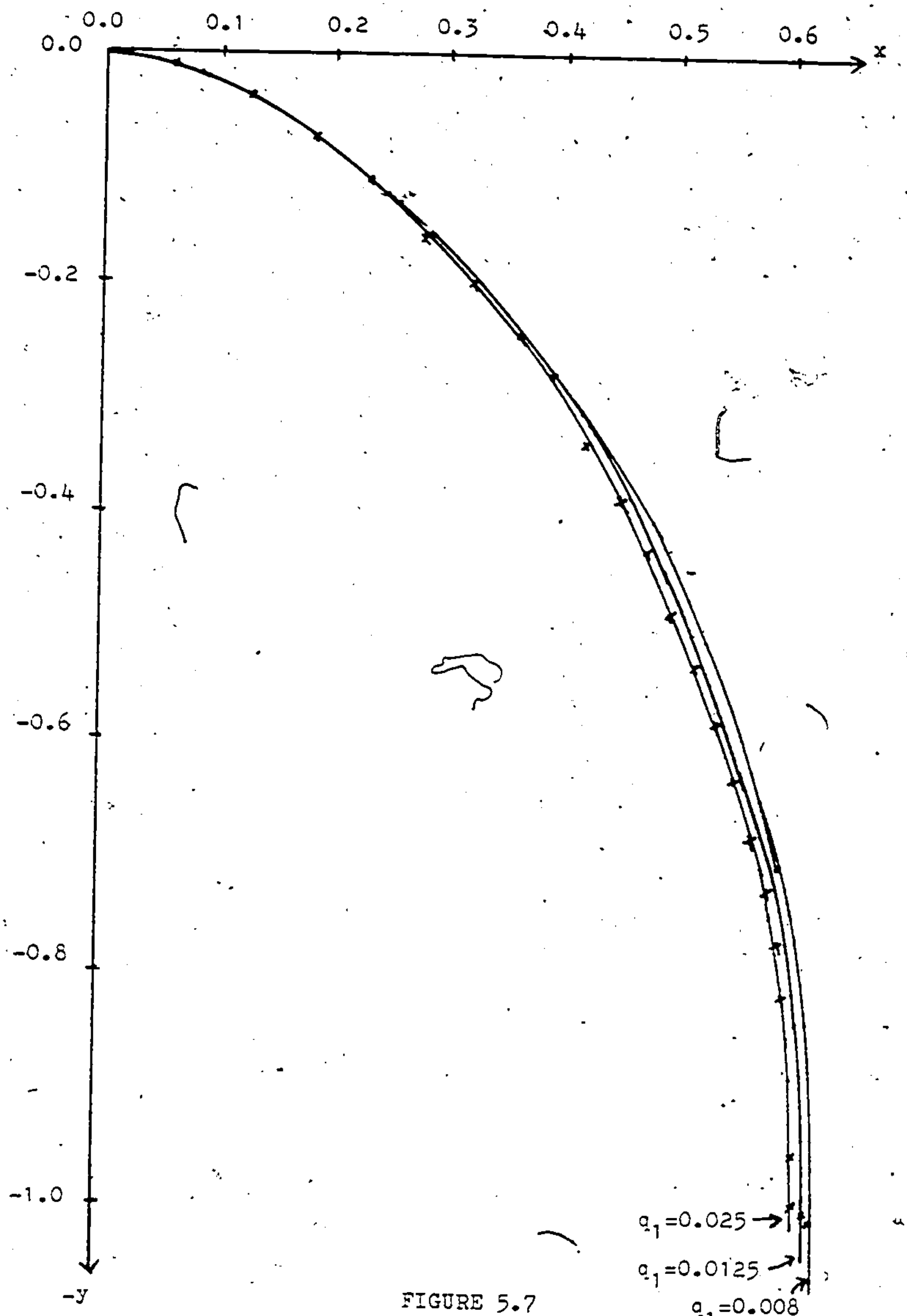


FIGURE 5.7

Shapes of jet near aperture E (when q_2=1.15)

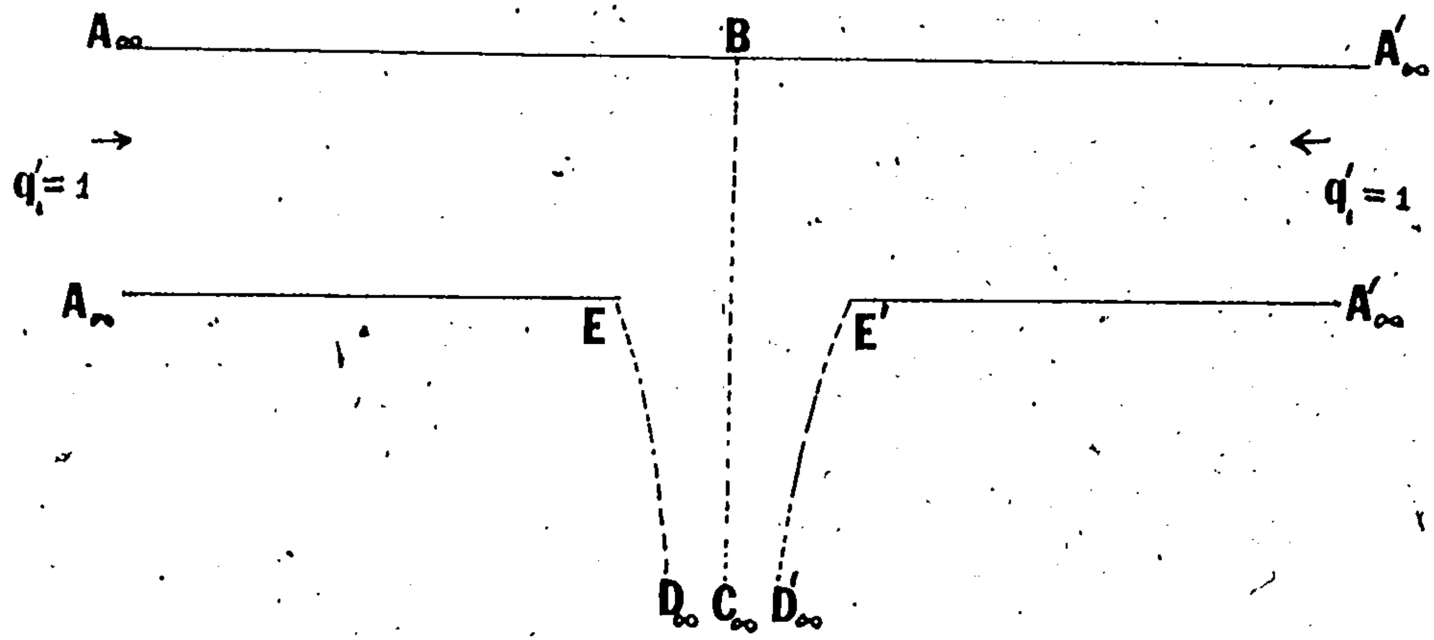


FIGURE 6.1a

The physical plane for a 2-D vertical jet from an infinite channel

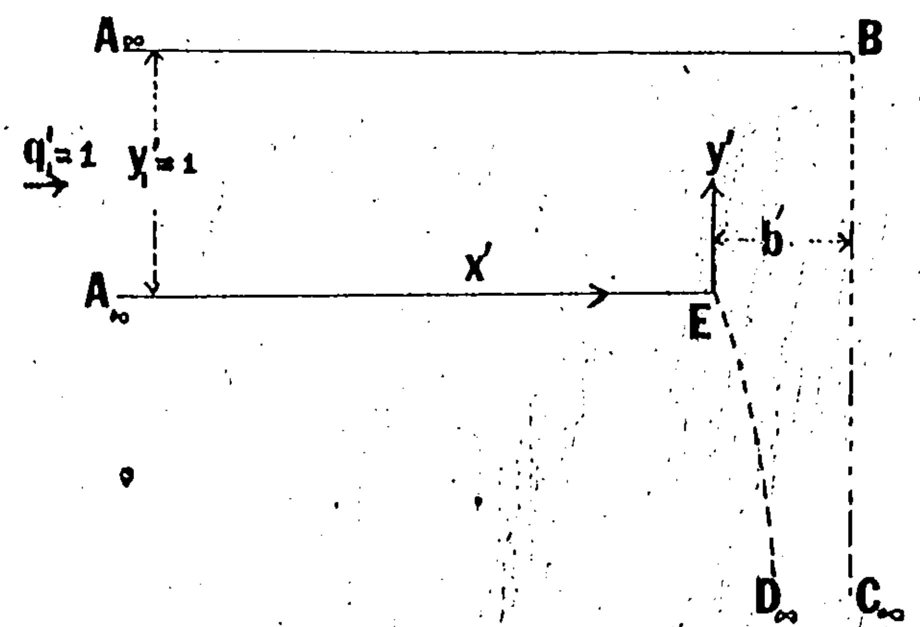


FIGURE 6. 1b

Half of the physical plane for a 2-D vertical jet from an infinite channel

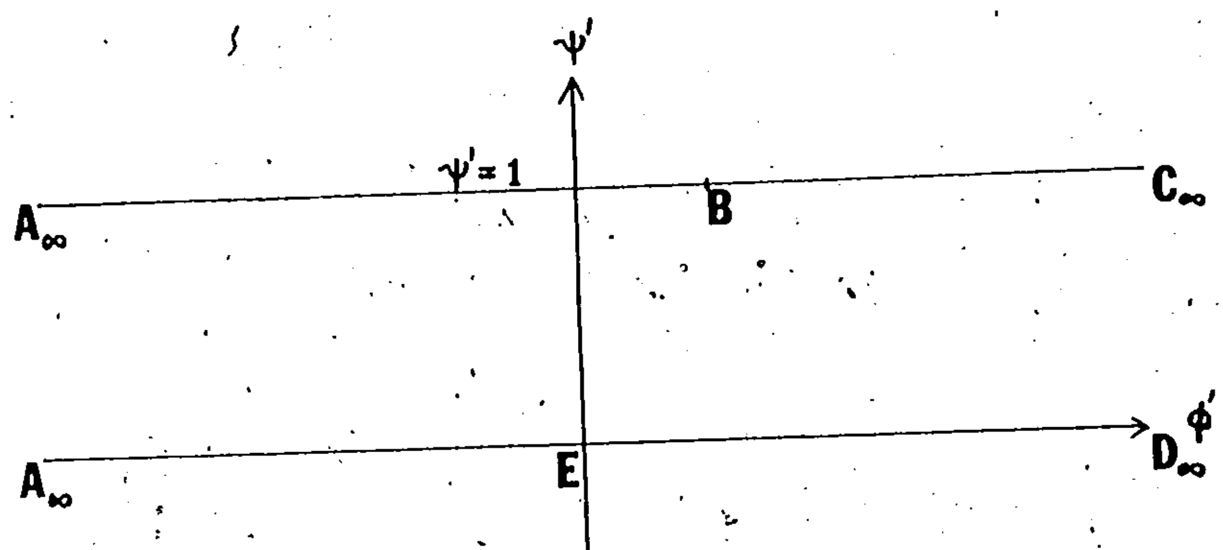


FIGURE 6. 2

The W'-plane for a 2-D vertical jet from an infinite channel

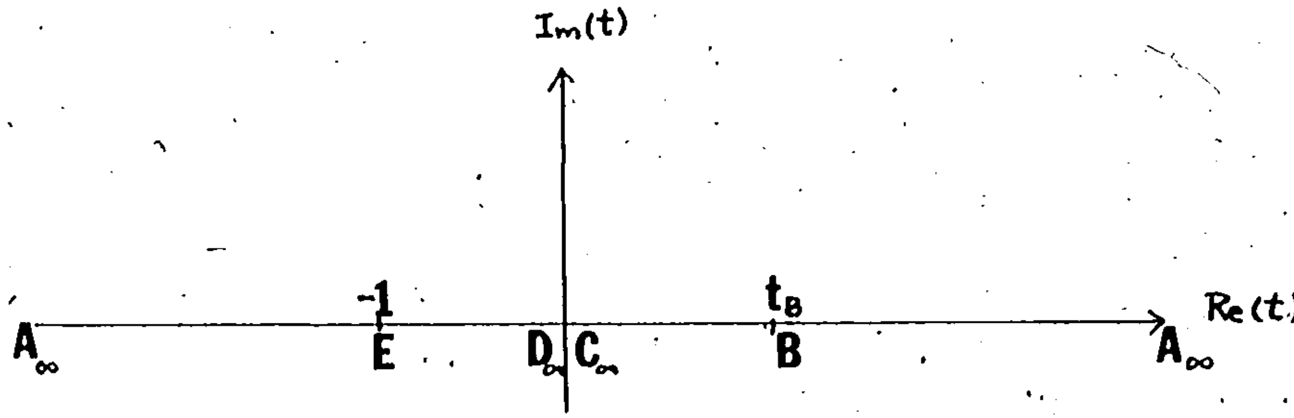


FIGURE 6.3

The t-plane for a 2-D vertical jet from an infinite channel

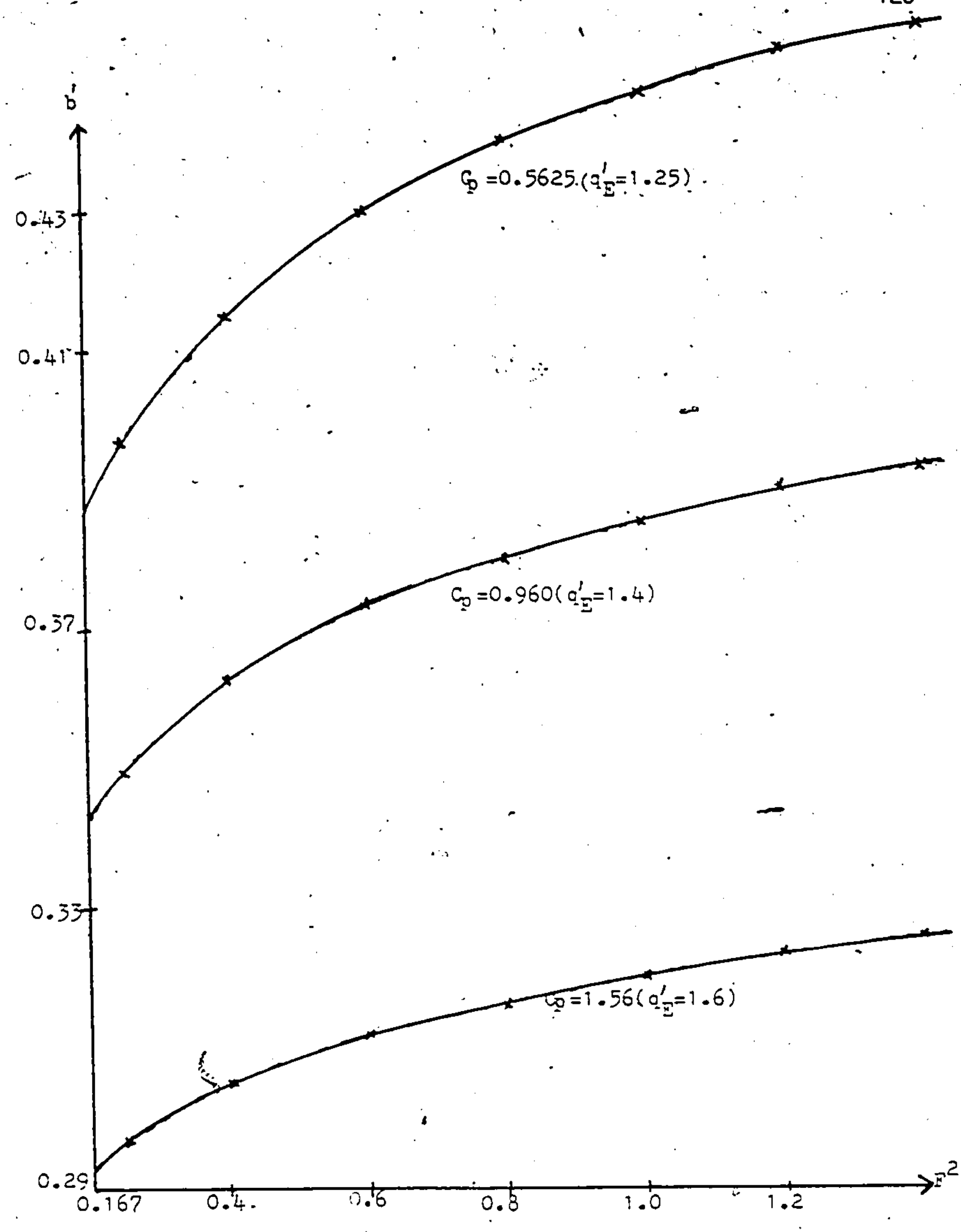


FIGURE 6. 4

b' vs F^2 for a 2-D vertical jet from an infinite channel

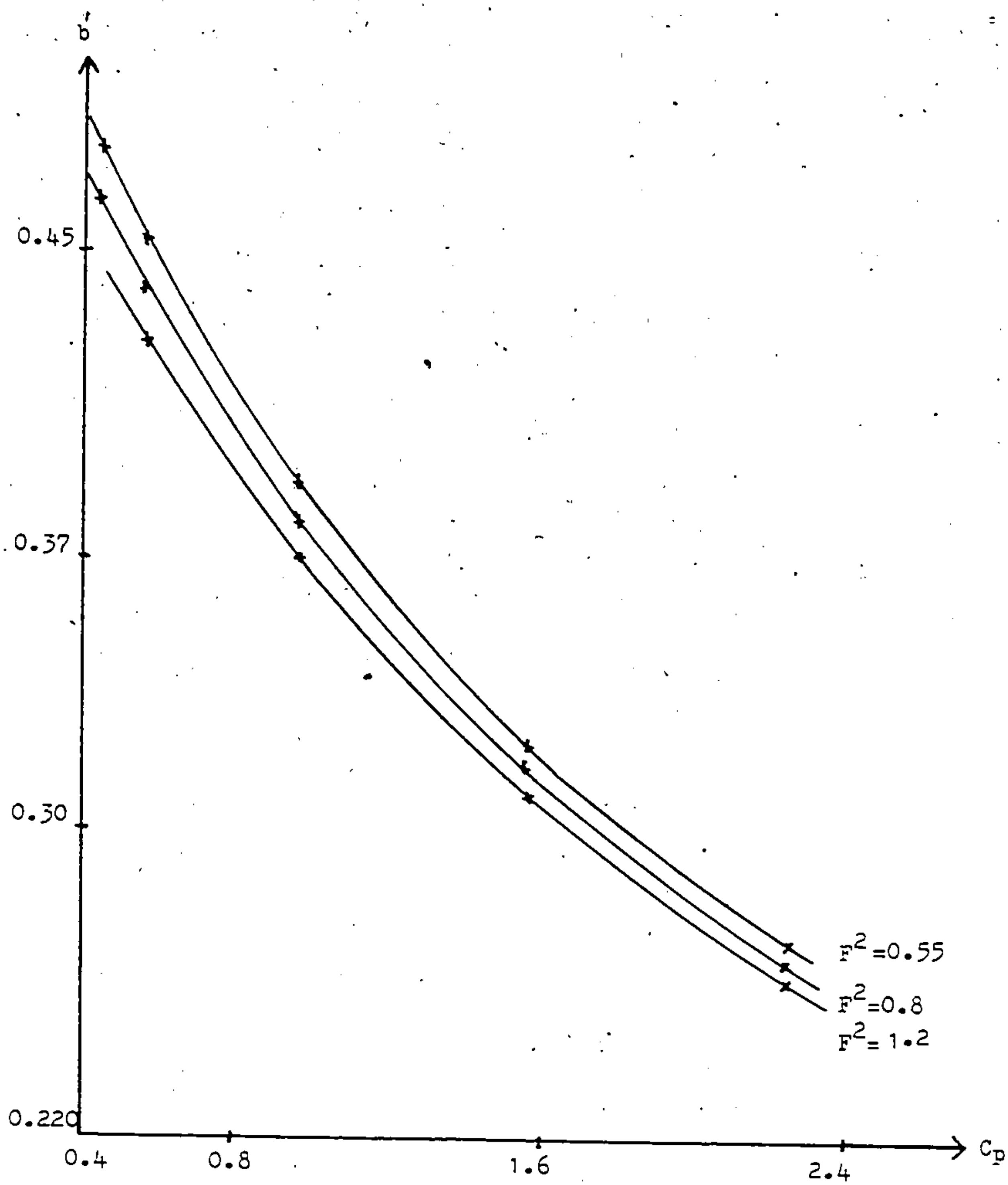


FIGURE 6. 5
b′ vs Cp for 2-D vertical jet from an infinite channel

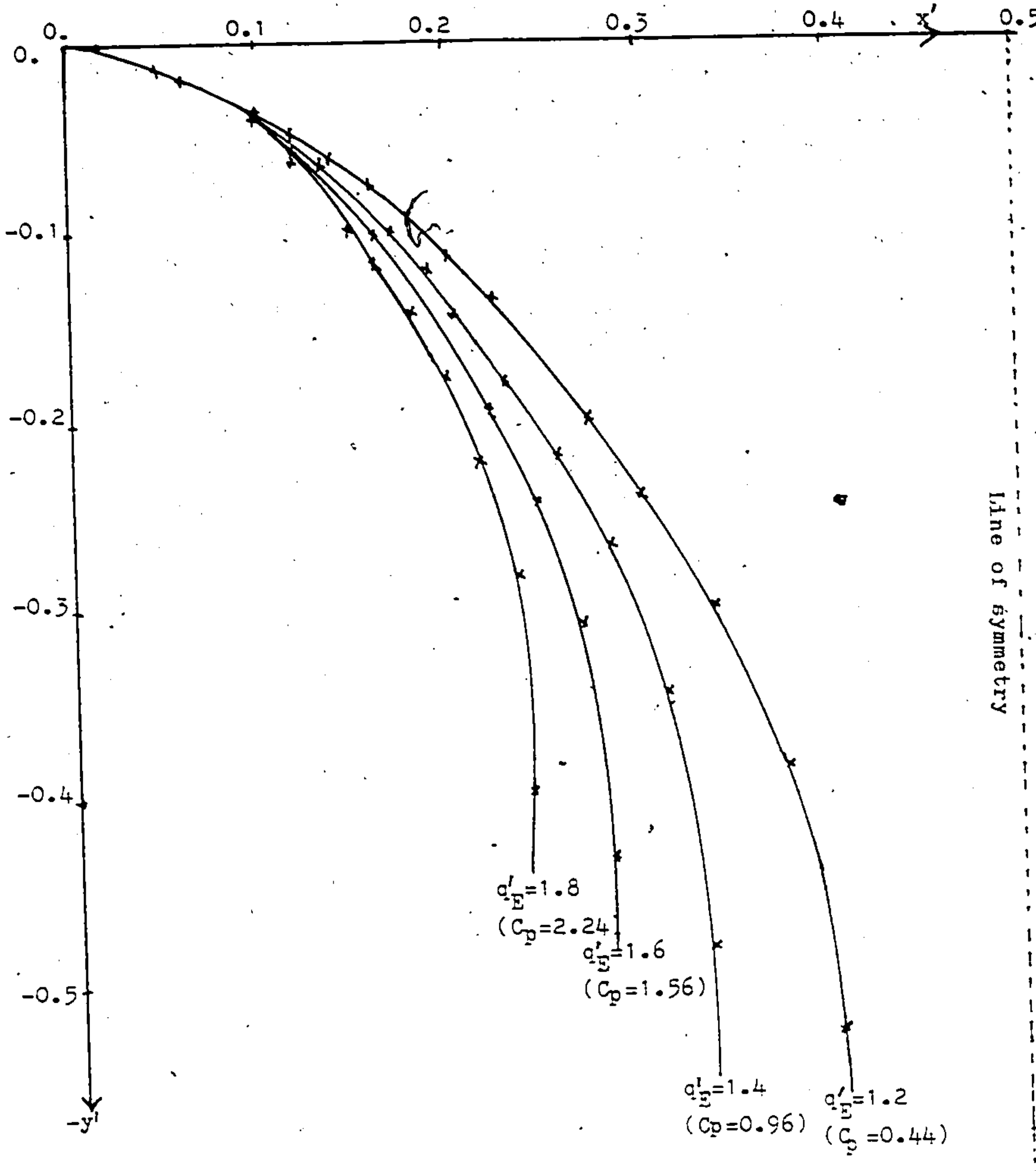


FIGURE 6. 6

Shapes of jet near the aperture E (when F^2=1.2)

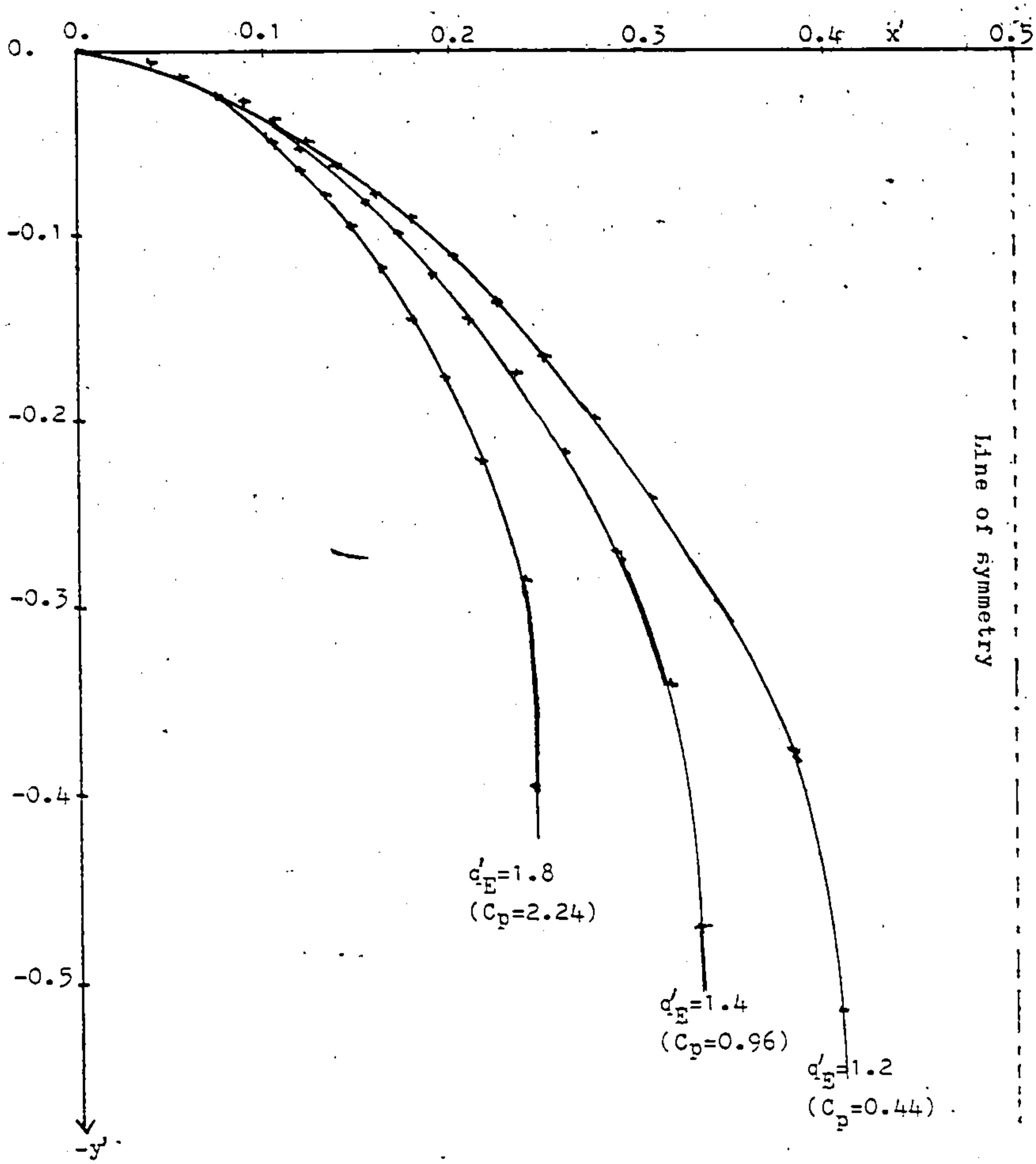


FIGURE 6.7

Shapes of jet near the aperture E (when $F^2=1.$)

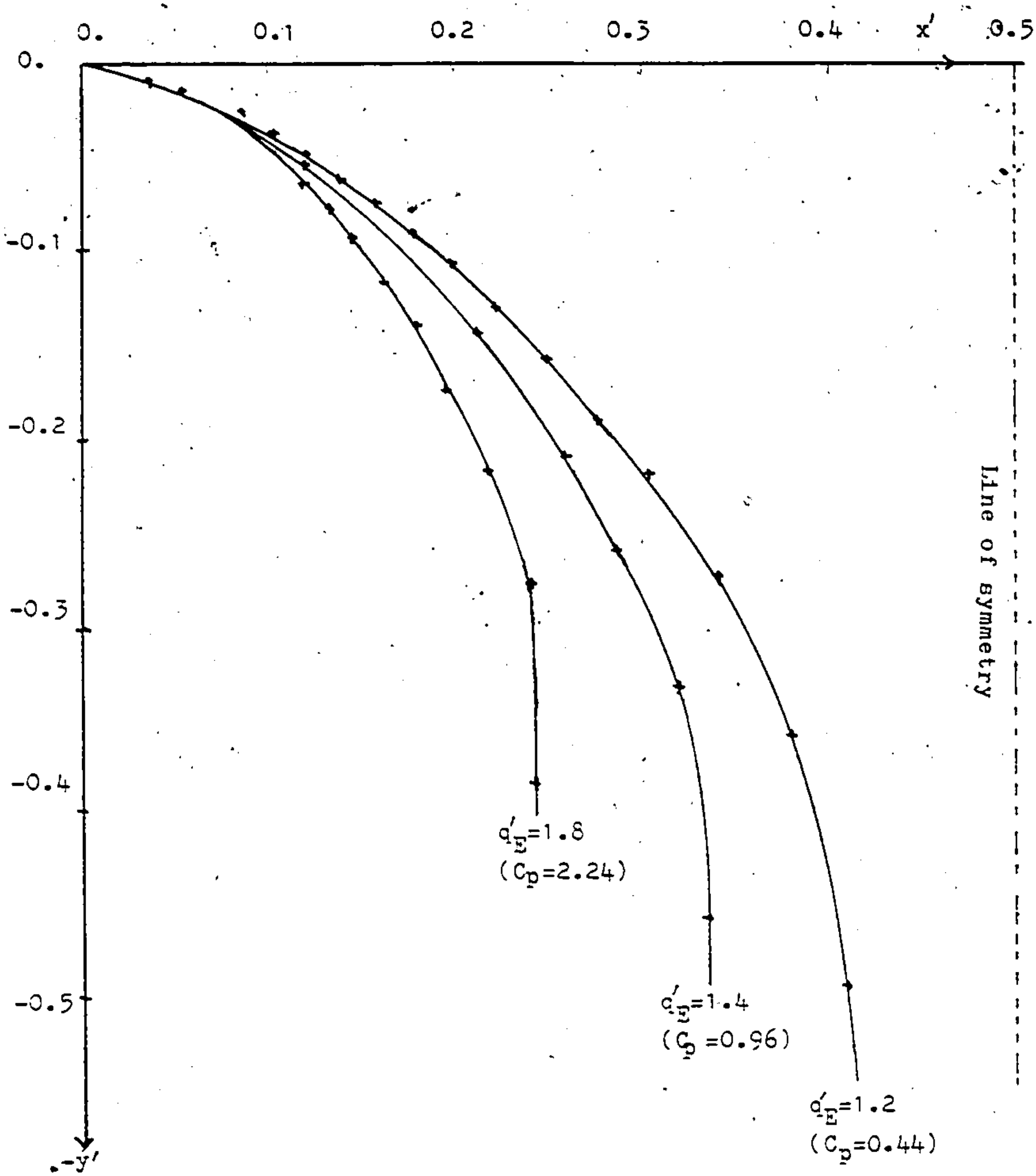


FIGURE 6. 8

Shapes of jet near the aperture E (when F^2=0.8)

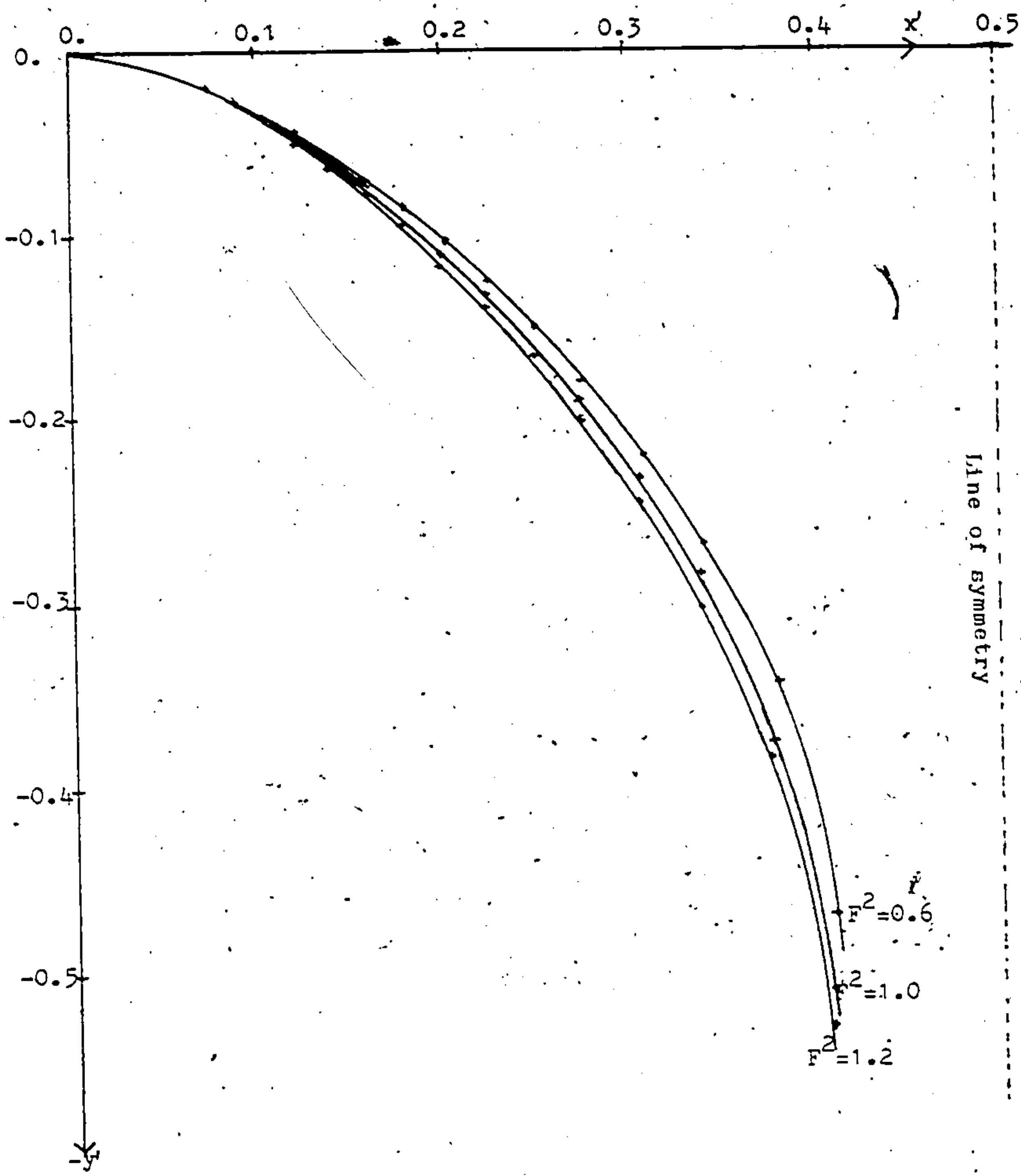


FIGURE 6. 9

Shapes of jet near the aperture Ξ (when $q'_E=1.2$ or $C_p=0.44$)

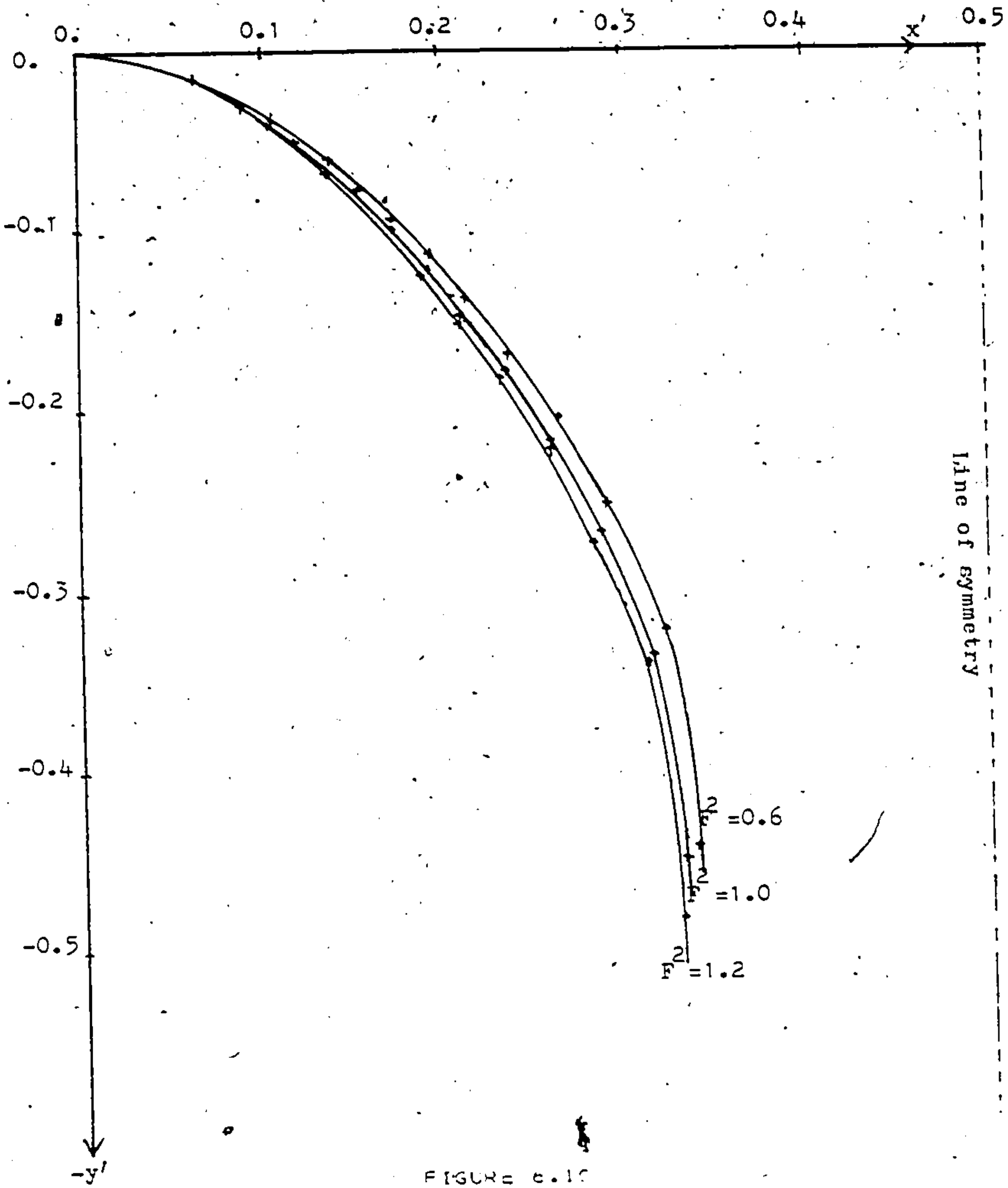


FIGURE 6.10

Shapes of jets near the aperture E (when $q'_E=1.4$ or $C_p=0.96$)

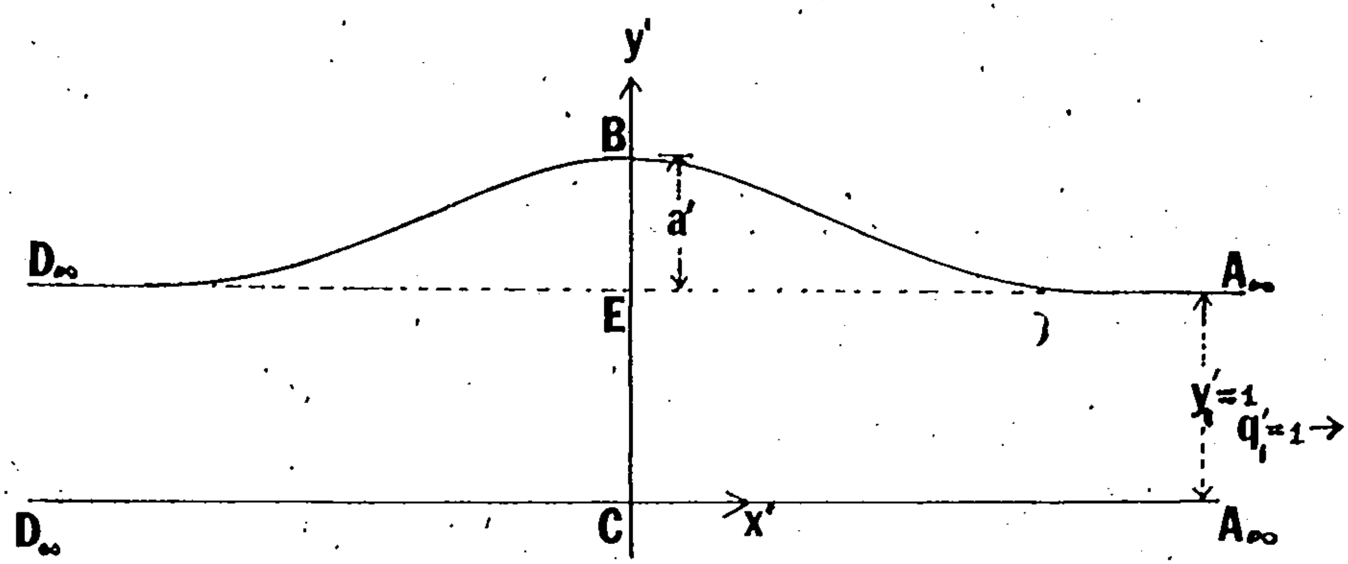


FIGURE 7. 1

The physical plane for the solitary wave

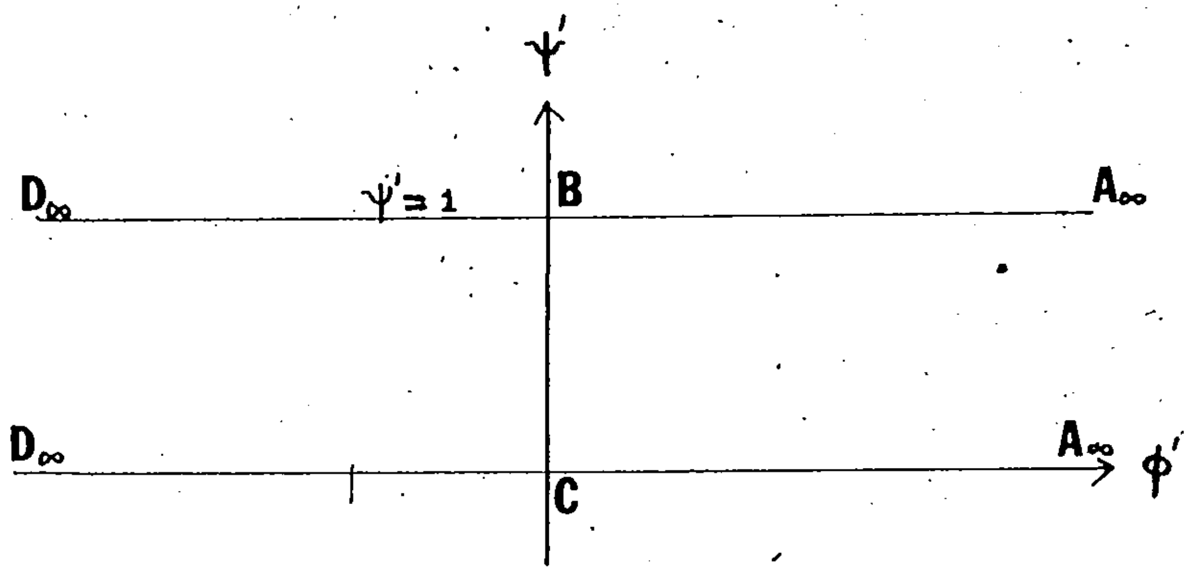


FIGURE 7. 2

The W'-plane for the solitary wave

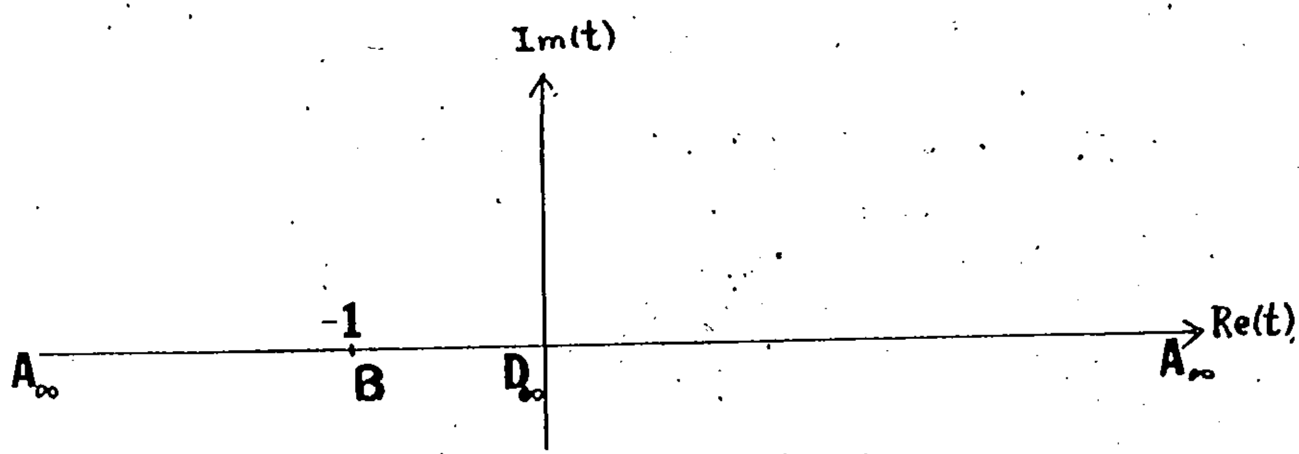


FIGURE 7. 3

The t-plane for the solitary wave

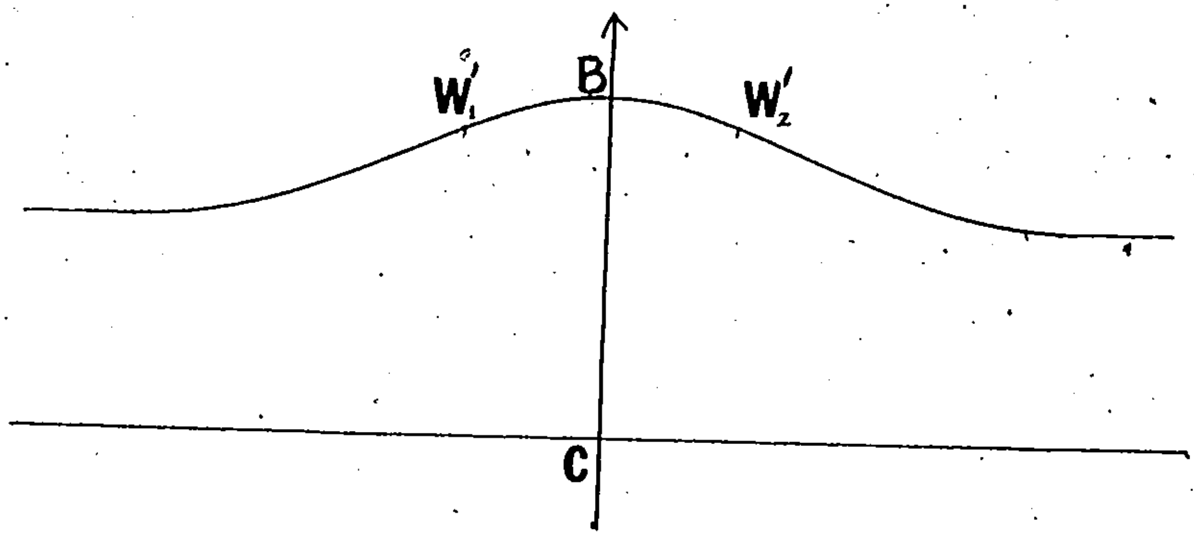


FIGURE 7. 4

The physical plane for the solitary wave

$Re(W_1)=-Re(W_2)$

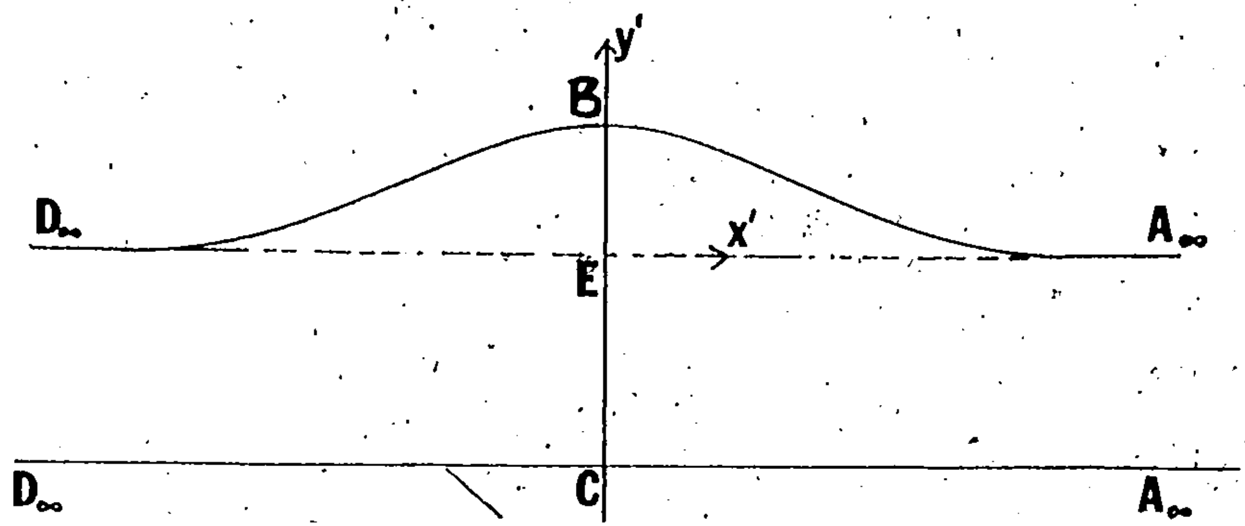


FIGURE 7.5

The physical plane for the solitary wave

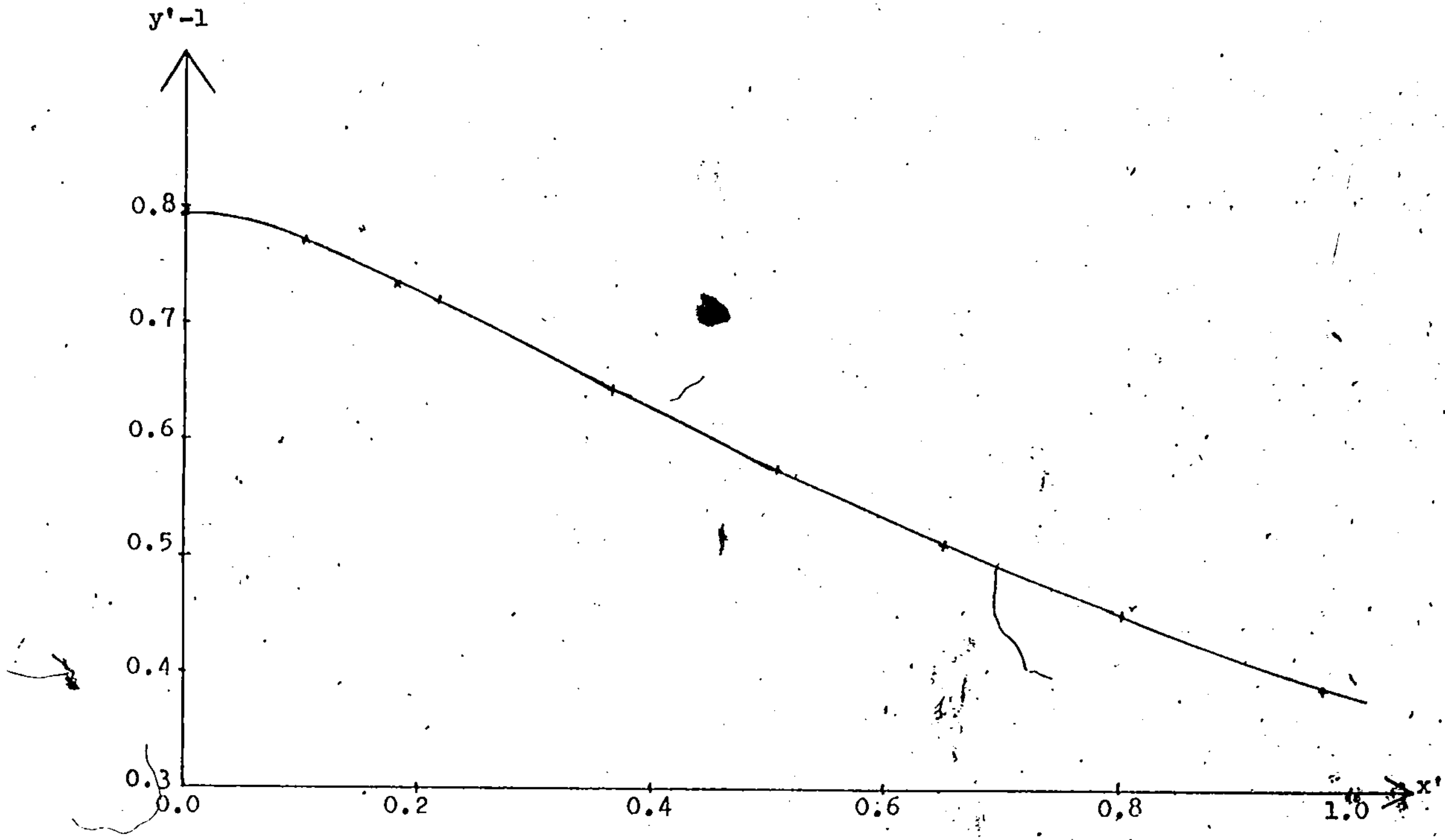


Figure 7.6 Shape of wave near crest when $q_B'=0.20$ and $N = 200$

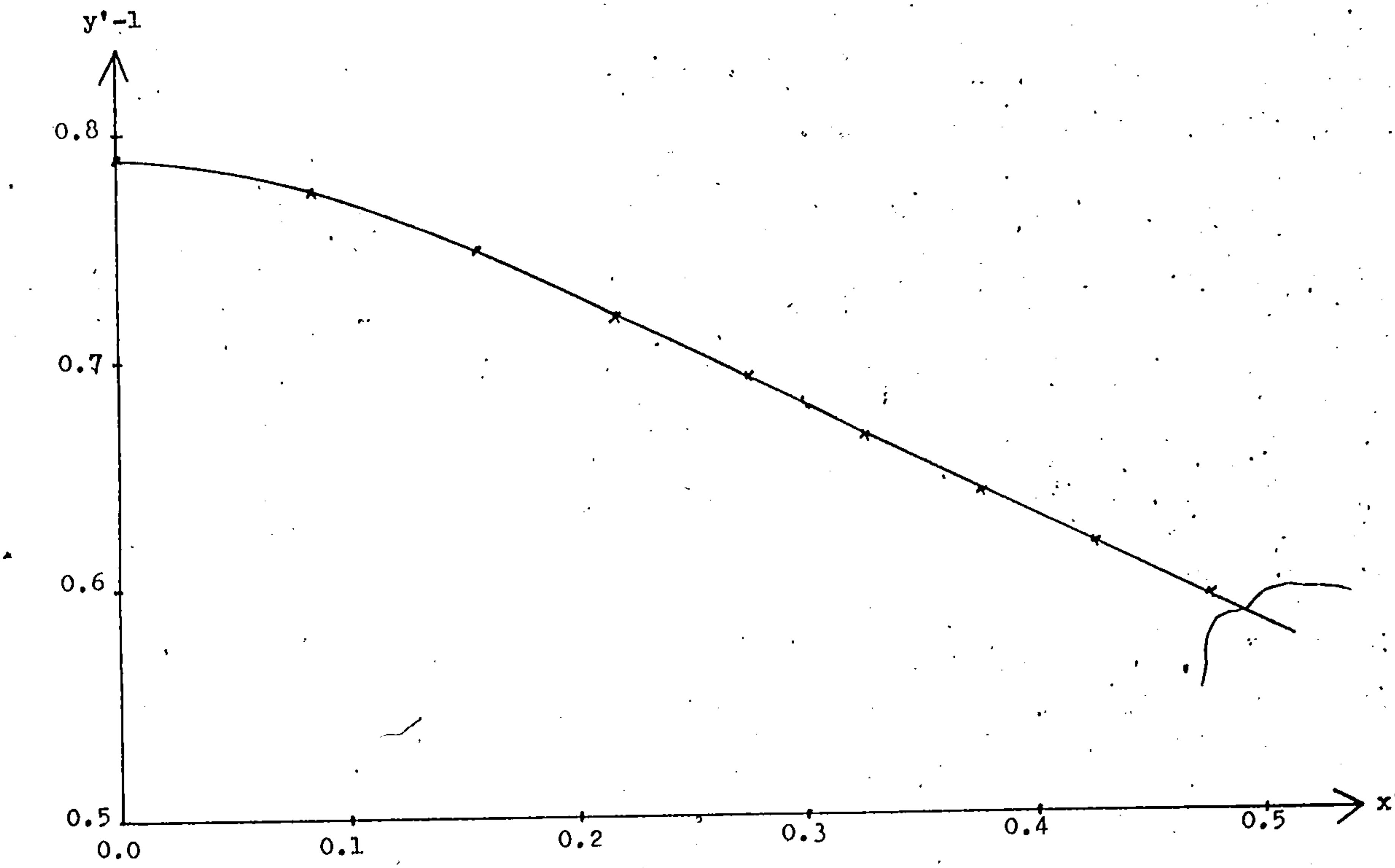


Figure 7.7 Shape of wave near crest when $q_B' = 0.222$ and $N = 600$ (the fastest wave)

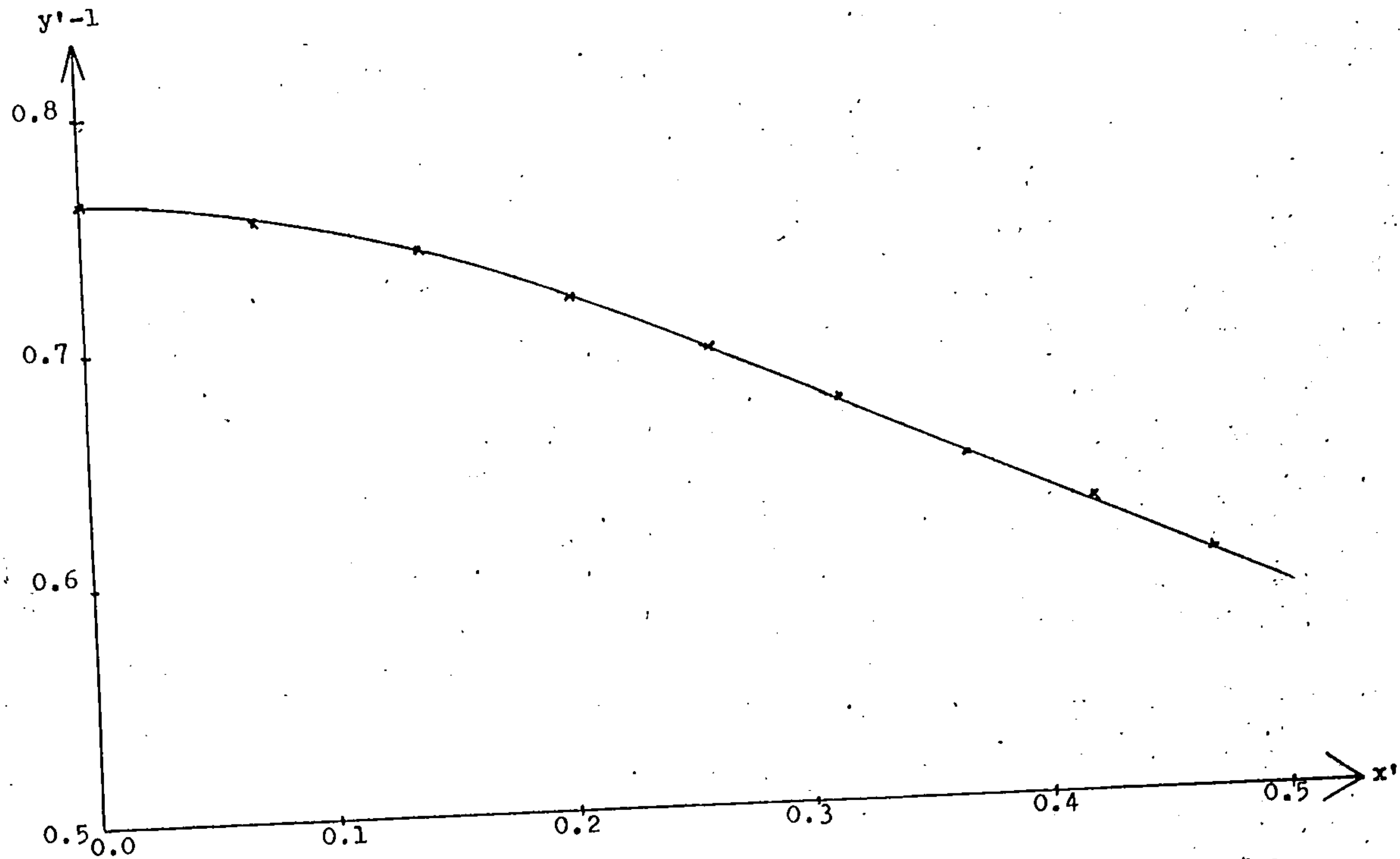


Figure 7.8 Shape of wave near crest when $q'_B = 0.28$ and $N = 600$

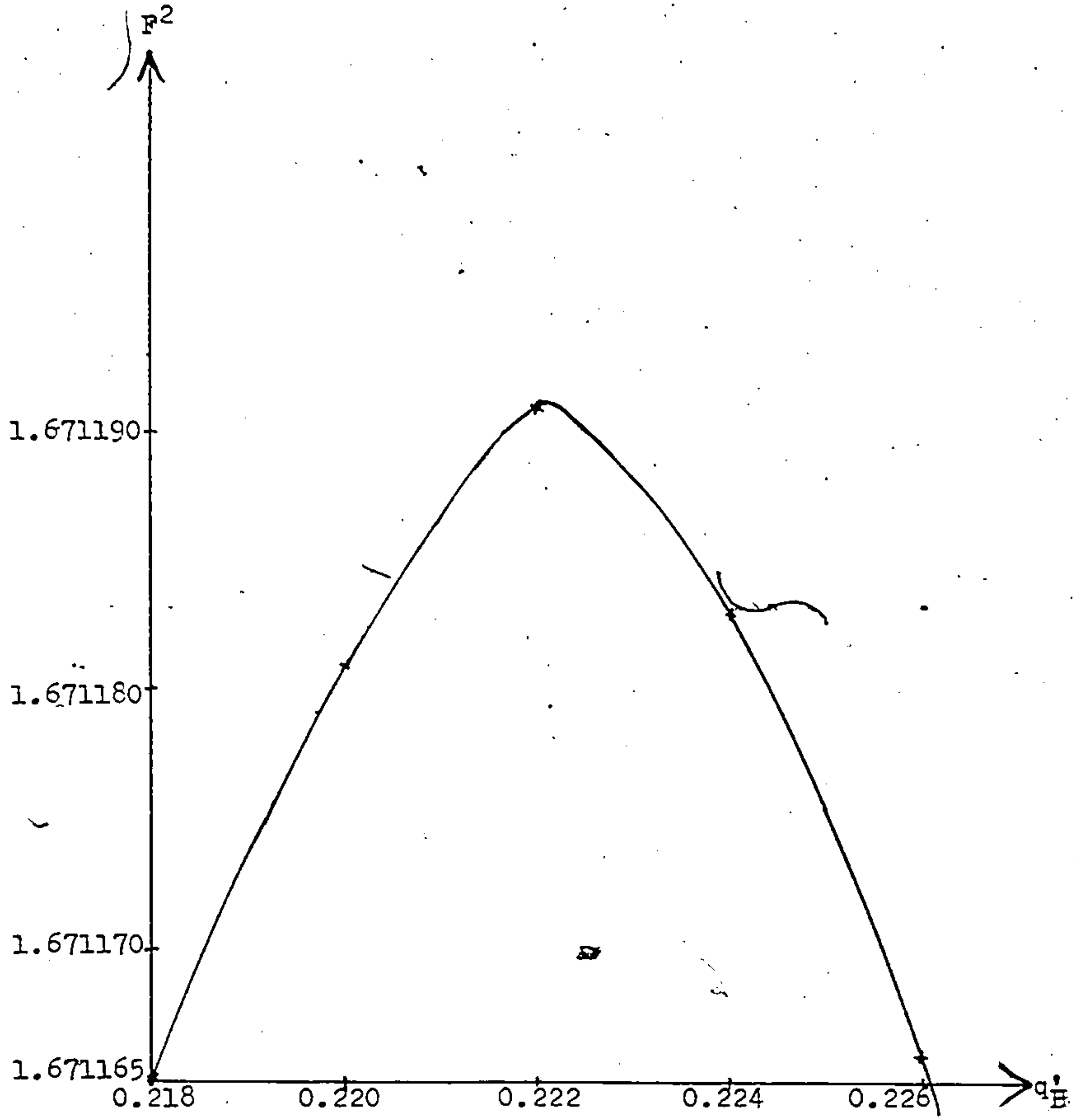


Figure 7.8 F^2 vs. q'_B for the solitary waves

APPENDIX A

The analytic continuation of harmonic functions across the free streamline is made possible by a method due to Lewy [31] based upon the following theorem which we quote in his notation:

Let $U(x,y)$ be harmonic near the origin in $y < 0$, and $V(x,y)$ a conjugate harmonic of U. Let U, V, U_x exist and be continuous in the semi-neighbourhood of $y \leq 0$ of the origin. If the boundary values on $y = 0$ satisfy a relation of the form

$$U_y = A(x, U, V, U_x), \tag{B}$$

in which A is an analytic function of all four arguments for all values occurring, then $U(x,y)$ and $V(x,y)$ are analytically extensible across $y = 0$.

We shall be concerned only with extension along the lines corresponding to x=constant (actually ϕ = constant in our notation). It suffices then to say that the analytic extension is given by [31, equation E_y]

$$\frac{dG(0,y)}{dy} = \frac{dF(0,y)}{dy} - \overset{*}{A}[iy,\ G(0,y) + F(0,y),\ i(G(0,y) - F(0,y)),$$
$$- i\cdot\frac{d}{dy}(G(0,y) + F(0,y))], \ldots \quad (E_y)$$

where

$$2F(z) = U(x,y) + iV(x,y),$$
$$2G(0,0) = U(0,0) - iV(0,0).$$

The analytic continuation of $F(z)$ along $x = 0$ is then given by

$$F(0,y) = \overline{G}(0, -y),\ y \geq 0,$$

where $G(0,y)$ is the solution of (E_y) subject to the initial condition $G(0,0) = U(0,0) - iV(0,0)$.

Changing to the notation of this paper, we replace x and y by ϕ and ψ and U and V by x and y respectively. The free surface condition can be written in the form (B) as

$$x_\psi = \left[\frac{1}{2g(h-y)} - x_\phi^2\right]^{1/2} = A\left[\phi, x, y, x_\phi\right], \tag{B_ψ}$$

where ϕ and x do not occur explicitly.

Then (E_y) assumes the form (along $\phi = 0$)

$$\frac{dG}{d\psi} = \frac{dF}{d\psi} - A[\phi, x, i(G-F), -i\frac{d}{d\psi}(G+F)],$$

which simplifies to

$$\frac{dG}{d\psi} = \frac{1}{\frac{dF}{d\psi}\ 8g[i(G-F)-h]}, \tag{E_ψ}$$

where now

$$F(0,\psi) = x(0,\psi) + iy(0,\psi),$$
$$F(0,\psi) = \overline{G}(0,-\psi) \qquad \psi \geq 0 .$$

Equation (E_ψ) was integrated numerically along $\phi = 0$ and $\phi = \phi_1$, providing continuations of x and y across the free streamlines. It was found that the difference between the continuations obtained using Lewy's method and those obtained using Shiffman's method [45] (i.e. $g = 0$) was quite small, and did not materially effect the shape of the wave except for an occasional change in the sixth decimal place.

APPENDIX B

```
$JOB     WATFIV   XXXXXXXXXX T.H.LIM
C
C        THE WATER WAVE PROBLEM
C        USING CAUCHY INTEGRAL EQUATION METHOD
C
C
         INTEGER MMMM(6)
         REAL*4 FX(29,2,2),FY(41,2,2)
         REAL*4 UX(29,2,4),UY(41,2,4),UV(6,169)
         REAL*4 VX(29,2,4),VY(41,2,4)
         REAL*4 WA(29),TV(29),QV(29),COSV(29),SINV(29)
         REAL*4 G(29),TT(29)
         REAL*4 X(29),Y(29),YA(29)
         REAL*4 TZ(41),QZ(41)
C
C
C
C        NY=NUMBER OF POINTS ON VERTTICAL SIDES
C        NX=NUMBER OF POINTS ON HORIZONTAL SIDES
C        HX=LENGTH OF SUBINTERVALS ON HORIZINTAL SIDES
C        HY=LENGTH OF SUBINTERVALS ON VERTICAL SIDES
C
C
         NX=29
         NY=41
         XX=0.43791
         YY=1.
         HX=XX/(NX-1)
         HY=YY/(NY-1)
         IC=6
C
C
C        SUBROUTINE XY---(X,Y) OF SIDES ON RECTANGLE
C        SUBROUTINE FREE---COMPUTE U ON FREE SURFACE USING
C                          THE FREE SURFACE CONDITION
C
C        SUBROUTINE ZERO2---INITALLY UX AND UY ARE ZERO
C        SUBROUTINE XTAN---COMPUTE DERIVATIVES OF U
C        SUBROUTINE XYTAN---COMPUTE DERIVATIVES OF U AND
C                           V ON 4 SIDES USING XTAN
C
C        SUBROUTINE DIFF---DIFF. EQUATION FROM LEWY'S METHOD
C        SUBROUTINE CUBI---LAGRANGIAN CUBIC INTERPOLATION
C                          FORMULA
C
C        SUBROUTINES CM,CP,EVEN AND EEM---COMPUTE U OR V ON
C                                  ONE SIDE OF RECTANGLE USING
C                                  CAUCHY INTEGRAL EQUATION
C
C        SUBROUTINE CNA---COMPUTE U AT THE CORNER A OR B
C                         USING CAUCHY INTEGRAL EQUATION
C
C        SUBROUTINE COMMU---COMPUTE U OR V ON 'ALL' SIDES
C                           AND CORNERS USING EEM AND CNA
C
C        SUBROUTINES  TRAPZ AND ZTRAP--- ARE IMPROVED
C                                        TRAPEZODAL RULES
C
C        SUBROUTINE PRINTG--- PRINT U AND V
C        SUBROUTINE PARA---COMPUTE THE PARAMETERS
C
C
```

```
      C
                CALL XY(FX,FY,NX,NY,HX,HY)
                Q0=1.
                Q1=0.8
      C
                CALL ZERO2(UX,VX,NX)
                CALL ZERO2(UY,VY,NY)
      C
      C
                UX(1,2,1)=ALOG(Q0)
                UX(NX,2,1)=ALOG(Q1)
                UY(NY,1,1)=UX(1,2,1)
                UY(NY,2,1)=UX(NX,2,1)
      C
                NXX=NX-1
                NYY=NY-1
                NZ=1
                DO 10 I=2,NYY


      10        NZ=NZ+I
      C
                FTRA=0.4
                FTRB=0.5
                FTRC=1.-FTRA-FTRB
                TPA=(Q0-Q1)/NZ*FTRA
                TPB=(Q0-Q1)/NZ*0.5*FTRB
                TPC=(Q0-Q1)/NXX*FTRC
      C
                DO 20 I=1,NYY
                J=NY-I
                UY(J,1,1)=ALOG(EXP(UY(J+1,1,1))-J*TPA)
      20        UY(J,2,1)=ALOG(EXP(UY(J+1,2,1))+2*J*TPB)
                DO 21 I=1,NX
      21        UX(I,1,1)=ALOG(EXP(UY(1,1,1))-(I-1)*TPC)
      C
                IW=2
                FACTOR=1.
                IR=NX-NX/(IW+1)*(IW+1)
                II=NX/(IW+1)*IW+IR
                JJ=II+1
                TEMP=3.141592654/6*(Q0-Q1)/(II-1)*FACTOR
      C
                DO 30 I=2,II
      30        VX(I,2,1)=-(I-1)*TEMP
                DO 31 I=JJ,NXX
      31        VX(I,2,1)=VX(II,2,1)+IW*(I+1-JJ)*TEMP
                PRINT,(VX(I,2,1),I=1,NX)
```

```
      C
            MMM=1
            CALL FREE(UX,VX,TT,TV,QV,WA,NX,HX,GRAV,TZ,QZ,MMM,NY)
            CALL XYTAN(UX,UY,NX,NY,HX,HY,VX,GRAV,TZ,QZ,MMM)
      C
            N1=NX
            N2=N1+NY
            N3=N2+NY
            N4=N3+NX
            MM=0
            PRINT99,MM
      99    FORMAT('1',I4)
      C
            DO 56 MMM=1,101
            MM=MM+1
            DO 5700 I=1,NX
            UV(MM,I)=EXP(UX(I,1,1))
            UV(MM,N3+I)=EXP(UX(I,2,1))
            UV(MM,N4+I)=-VX(I,2,1)
      5700  CONTINUE
      C
            DO 5701 I=1,NY
            UV(MM,N1+I)=EXP(UY(I,2,1))
            UV(MM,N2+I)=EXP(UY(I,1,1))
      5701  CONTINUE
      C
            IF(MM.NE.IC) GO TO 58
            CALL PRINTG(UV,1,10,MM1,IC,3*NX+2*NY,MMMM)
            MM=0
            IF(MMM.LT.2*IC.OR.MMM.GT.5*IC) GO TO 58
            CALL AVE(UV,IC,3*NX+2*NY)
      C
            DO 57 I=1,NX
            UX(I,1,1)=ALOG(UV(IC,I))
            UX(I,2,1)=ALOG(UV(IC,N3+I))
      57    VX(I,2,1)=-UV(IC,N4+I)
            DO 570 I=1,NY
            UY(I,2,1)=ALOG(UV(IC,N1+I))
      570   UY(I,1,1)=ALOG(UV(IC,N2+I))
      C
            CALL XYTAN(UX,UY,NX,NY,HX,HY,VX,GRAV,TZ,QZ,MMM)
            CALL FREE(UX,VX,TT,TV,QV,WA,NX,HX,GRAV,TZ,QZ,MMM,NY)
      58    CONTINUE
      C
            CALL COMUU(UX,UY,VX,VY,FX,FY,NX,NY,HX,HY)
            CALL XYTAN(UX,UY,NX,NY,HX,HY,VX,GRAV,TZ,QZ,MMM)
            CALL FREE(UX,VX,TT,TV,QV,WA,NX,HX,GRAV,TZ,QZ,MMM,NY)
            IF(MMM.GE.20) GO TO 80
            IF(MMM/5*5.NE.MMM) GO TO 56
            GO TO 85
      80    IF(MMM/3*3.NE.MMM) GO TO 56
      C
      85    PRINT99,MMM
            PRINT93,Q0,Q1,NX,NY
            PRINT,' '
            PRINT94,XX,YY,HX,HY
      93    FORMAT(3X,'Q0=',F10.5,5X,'Q1=',F10.5,5X,'NX=',
           1       I10,5X,'NY=',I10)
      94    FORMAT(3X,'XX=',F10.5,5X,'YY=',F10.5,5X,'HX=',
           1       F10.5,5X,'HY=',F10.5)
            CALL PARA(UX,VX,TT,TV,QV,X,Y,YA,NX,HX,XX,GRAV,
           1       Q,COSV,SINV,UY,NY,HY,TZ)
      C
      56    CONTINUE
      C
            STOP
            END
```

```
      SUBROUTINE ZERO2(UX,VX,NX)
      REAL*4 UX(NX,2,4),VX(NX,2,4)
      DO 10 I=1,NX
      DO 10 J=1,2
      DO 10 K=1,4
      UX(I,J,K)=0.
      VX(I,J,K)=0.
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE AVE(UV,IC,KY)
      REAL*4 UV(IC,KY)
      DO 40 J=1,KY
      SUM=UV(1,J)
      DO 20 I=2,IC
   20 SUM=SUM+UV(I,J)
   40 UV(IC,J)=SUM/IC
      RETURN
      END
```

```
      SUBROUTINE XY(FX,FY,NX,NY,HX,HY)
      REAL*4 FX(NX,2,2),FY(NY,2,2)
      XX=HX*(NX-1)
      YY=HY*(NY-1)
      DO 13 I=1,NX
      FX(I,1,1)=(I-1)*HX
      FX(I,1,2)=0.
      FX(I,2,1)=(I-1)*HX
   13 FX(I,2,2)=YY
C
      DO 24 I=1,NY
      FY(I,1,1)=0.
      FY(I,1,2)=(I-1)*HY
      FY(I,2,1)=XX
   24 FY(I,2,2)=(I-1)*HY
      RETURN
      END
```

```
      SUBROUTINE XTAN(U,NX,HX,K,III,JJJ,GRAV,TZ,QZ,MMM,NY)
C
      REAL*4 DYSTAR(4),TZ(NY),QZ(NY)
      REAL*4 U(NX,2,4)
C
      FT(A,B,C,D,E,F)=(45*(A-B)-9*(C-D)+(E-F))/(60*HX)
      GT(A,B,C,D,E,F,G)=(270*(A+B)-27*(C+D)+2*(E+F)-490*G)/
     1                  (180*HX**2)
      HT(A,B,C,D,E,F)=(-13*(A-B)+8*(C-D)-(E-F))/(8*HX**3)
C
      UA=III*U(2,K,1)
      UB=III*U(3,K,1)
      UC=III*U(4,K,1)
C
      KK=NX-3
      DO 10 I=4,KK
      U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1            U(I-2,K,1),U(I+3,K,1),U(I-3,K,1))
      U(I,K,3)=GT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1            U(I-2,K,1),U(I+3,K,1),U(I-3,K,1),
     1            U(I,K,1))
   10 U(I,K,4)=HT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1            U(I-2,K,1),U(I+3,K,1),U(I-3,K,1))
C
      U(1,K,2)=FT(U(2,K,1),UA,U(3,K,1),UB,U(4,K,1),UC)
      U(1,K,3)=GT(U(2,K,1),UA,U(3,K,1),UB,U(4,K,1),UC,
     1            U(1,K,1))
      U(1,K,4)=HT(U(2,K,1),UA,U(3,K,1),UB,U(4,K,1),UC)
C
      U(2,K,2)=FT(U(3,K,1),U(1,K,1),U(4,K,1),UA,U(5,K,1),
     1            UB)
      U(2,K,3)=GT(U(3,K,1),U(1,K,1),U(4,K,1),UA,U(5,K,1),
     1            UB,U(2,K,1))
      U(2,K,4)=HT(U(3,K,1),U(1,K,1),U(4,K,1),UA,U(5,K,1),
     1            UB)
C
      U(3,K,2)=FT(U(4,K,1),U(2,K,1),U(5,K,1),U(1,K,1),
     1       U(6,K,1),UA)
      U(3,K,3)=GT(U(4,K,1),U(2,K,1),U(5,K,1),U(1,K,1),
     1       U(6,K,1),UA,U(3,K,1))
      U(3,K,4)=HT(U(4,K,1),U(2,K,1),U(5,K,1),U(1,K,1),
     1       U(6,K,1),UA)
C
```

```
C
      IF(JJJ.EQ.-1) GO TO 20
      VA=III*U(NX-1,K,1)
      VB=III*U(NX-2,K,1)
      VC=III*U(NX-3,K,1)
      GO TO 40
20    IF(MMM.GE.10) GO TO 30
      QQ=EXP(U(NX,K,1))**2
      VA=ALOG(QQ/EXP(U(NX-1,K,1)))
      VB=ALOG(QQ/EXP(U(NX-2,K,1)))
      VC=ALOG(QQ/EXP(U(NX-3,K,1)))
      GO TO 40
30    CALL DIFF(U,NX,HX,DYSTAR,GRAV,K,TZ,BZ)
      VA=ALOG(1./DYSTAR(2))
      VB=ALOG(1./DYSTAR(3))
      VC=ALOG(1./DYSTAR(4))
40    CONTINUE
C
      I=NX-2
      U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1           U(I-2,K,1),VA,U(I-3,K,1))
      U(I,K,3)=GT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1           U(I-2,K,1),VA,U(I-3,K,1),U(I,K,1))
      U(I,K,4)=HT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1           U(I-2,K,1),VA,U(I-3,K,1))
C
      I=NX-1
      U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),VA,U(I-2,K,1),
     1           VB,U(I-3,K,1))
      U(I,K,3)=GT(U(I+1,K,1),U(I-1,K,1),VA,U(I-2,K,1),
     1           VB,U(I-3,K,1),U(I-3,K,1))
      U(I,K,4)=HT(U(I+1,K,1),U(I-1,K,1),VA,U(I-2,K,1),
     1           VB,U(I-3,K,1))
C
      U(NX,K,2)=FT(VA,U(NX-1,K,1),VB,U(NX-2,K,1),VC,
     1           U(NX-3,K,1))
      U(NX,K,3)=GT(VA,U(NX-1,K,1),VB,U(NX-2,K,1),VC,
     1           U(NX-3,K,1),U(NX,K,1))
      U(NX,K,4)=HT(VA,U(NX-1,K,1),VB,U(NX-2,K,1),VC,
     1           U(NX-3,K,1))
      RETURN
      END
```

```
      SUBROUTINE XYTAN(UX,UY,NX,NY,HX,HY,VX,GRAV,TZ,QZ,MMM)
      REAL*4 TZ(NY),QZ(NY)
      REAL*4 UX(NX,2,4),UY(NY,2,4)
      REAL*4 VX(NX,2,4)
      CALL XTAN(UX,NX,HX,1,+1,+1,GRAV,TZ,QZ,MMM,NY)
      CALL XTAN(VX,NX,HX,2,-1,+1,GRAV,TZ,QZ,MMM,NY)
      CALL XTAN(UY,NY,HY,1,+1,-1,GRAV,TZ,QZ,MMM,NY)
      CALL XTAN(UY,NY,HY,2,+1,-1,GRAV,TZ,QZ,MMM,NY)
      RETURN
      END


      SUBROUTINE DIFF(UY,NY,HY,DYSTAR,GRAV,K,TZ,QZ)
      REAL*4 UY(NY,2,4),TZ(NY),QZ(NY)
      REAL*4 Q(4),QH(3),Y(4),YH(3),YSTAR(4),DYSTAR(4)
      YT(Y,YSTAR,Q)=Q/(QQ**2+2*GRAV*H-GRAV*(Y+YSTAR))
      KK=NY-4
      DO 10 I=1,NY
   10 TZ(I)=1./EXP(UY(I,K,1))
      DO 20 I=1,4
   20 CALL ZTRAP(TZ,KK+I,HY,Y(5-I),QZ)
C
      DO 30 I=1,4
   30 Q(I)=EXP(UY(NY-I+1,K,1))
      H=Y(1)
      QQ=Q(1)
      CALL CUBI(Q,QH)
      CALL CUBI(Y,YH)
C
      YSTAR(1)=Y(1)
      DO 40 I=1,3
      DYSTAR(I)=YT(Y(I),YSTAR(I),Q(I))
      HKA=HY*DYSTAR(I)
      HKB=HY*YT(YH(I),YSTAR(I)+0.5*HKA,QH(I))
      HKC=HY*YT(YH(I),YSTAR(I)+0.5*HKB,QH(I))
      HKD=HY*YT(Y(I+1),YSTAR(I)+HKC,Q(I+1))
   40 YSTAR(I+1)=YSTAR(I)+(HKA+2*HKB+2*HKC+HKD)/6.
      DYSTAR(4)=YT(Y(4),YSTAR(4),Q(4))
      RETURN
      END


      SUBROUTINE CUBI(Q,QH)
      REAL*4 Q(4),QH(3)
      Y(A)=-(A-2)*(A-3)*(A-4)/6*YA+0.5*(A-1)*(A-3)*(A-4)*YB-
     1     0.5*(A-1)*(A-2)*(A-4)*YC+(A-1)*(A-2)*(A-3)*YD/6.
      YA=Q(1)
      YB=Q(2)
      YC=Q(3)
      YD=Q(4)
      QH(1)=Y(1.5)
      QH(2)=Y(2.5)
      QH(3)=Y(3.5)
      RETURN
      END
```

```
      SUBROUTINE CM(UY,VY,FX,FY,NX,NY,HY,I,K,L,III,IND,SUM)
C
      REAL*4 UY(NY,2,4),VY(NY,2,4)
      REAL*4 FX(NX,2,2),FY(NY,2,2)
C
      FNAA(RO,SO,R,S)=0.5*ALOG((RO-R)**2+(SO-S)**2)
      FNXX(RO,SO,R,S)=ABS((S-SO)/((RO-R)**2+(SO-S)**2))
      FNYY(RO,SO,R,S)=ABS((R-RO)/((RO-R)**2+(SO-S)**2))
      SIMP(A,B,C,D,E,F,H)=0.3333333333*(A*D+B*E+4*C*F)*H
C
      NYY=NY-1
      IF(L.EQ.2.AND.III.EQ.-1) GO TO 20
      IF(IND.EQ.-1) GO TO 30
      DO 25 J=2,NYY,2
      X=FNAA(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      Y=FNAA(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      Z=FNAA(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
   25 SUM=SUM+IND*III*SIMP(VY(J+1,L,2),VY(J-1,L,2),
     1              VY(J,L,2),X,Y,Z,HY)
      RETURN
   30 IF(L.EQ.1.AND.III.EQ.+1) GO TO 40
      DO 35 J=2,NYY,2
      U=FNYY(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      V=FNYY(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      W=FNYY(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
   35 SUM=SUM+SIMP(UY(J+1,L,1),UY(J-1,L,1),UY(J,L,1),
     1              U,V,W,HY)
      RETURN
C
   40 DO 45 J=2,NYY,2
      U=FNXX(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      V=FNXX(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      W=FNXX(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
   45 SUM=SUM+SIMP(UY(J+1,L,1),UY(J-1,L,1),UY(J,L,1),
     1              U,V,W,HY)
      RETURN
C
C
   20 DO 80 J=2,NYY,2
      X=FNAA(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      Y=FNAA(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      Z=FNAA(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
      U=FNXX(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      V=FNXX(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      W=FNXX(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
   80 SUM=SUM+IND*III*SIMP(VY(J+1,L,2),VY(J-1,L,2),
     1              VY(J,L,2),X,Y,Z,HY)+
     1    SIMP(UY(J+1,L,1),UY(J-1,L,1),UY(J,L,1),U,V,W,HY)
      RETURN
      END
```

```
      SUBROUTINE CP(UY,VY,FX,FY,NX,NY,HY,I,K,L,III,IND,
     1              SUM,IA,IB,IAB,H)
C
      REAL*4 UY(NY,2,4),VY(NY,2,4)
      REAL*4 FX(NX,2,2),FY(NY,2,2)
C
      FNAA(RO,SO,R,S)=0.5*ALOG((RO-R)**2+(SO-S)**2)
      FNXX(RO,SO,R,S)=ABS((S-SO)/((RO-R)**2+(SO-S)**2))
      FNYY(RO,SO,R,S)=ABS((R-RO)/((RO-R)**2+(SO-S)**2))
      SIMP(A,B,C,D,E,F,H)=0.3333333333*(A*D+B*E+4*C*F)*H
      SIMZ(F,A,B,C,X,Y,Z)=F*ANGLE+A*X+B*Y+C*Z
      FT(A,B,C,D)=(270*A-27*B+2*C-245*D)/180.
      GT(A,B,C,D)=(-39*A+12*B-C+28*D)/72.
      HT(A,B,C,D)=(15*A-6*B+C-10*D)/360.
C
      IF(I/2*2.NE.I) GO TO 5
      HX=H
      GO TO 6
 5    HX=2*H
 6    CONTINUE
      NYY=NY-1
      ANGLE=ATAN2(2*HY,HX)
      IF(.NOT.(L.EQ.2.AND.III.EQ.-1)) GO TO 70
      DO 25 J=2,NYY,2
      X=FNAA(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      Y=FNAA(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      S=FNAA(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
 25   SUM=SUM+IND*III*SIMP(VY(J+1,L,2),VY(J-1,L,2),
     1              VY(J,L,2),X,Y,S,HY)
 26   DO 20 J=IA,IB,2
      U=FNXX(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      V=FNXX(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      W=FNXX(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
 20   SUM=SUM+SIMP(UY(J+1,L,1),UY(J-1,L,1),UY(J,L,1),
     1              U,V,W,HY)
      GO TO 90
 70   IF(L.EQ.1.AND.III.EQ.+1) GO TO 26
      DO 10 J=IA,IB,2
      U=FNYY(FX(I,K,1),FX(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      V=FNYY(FX(I,K,1),FX(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      W=FNYY(FX(I,K,1),FX(I,K,2),FY(J,L,1),FY(J,L,2))
 10   SUM=SUM+SIMP(UY(J+1,L,1),UY(J-1,L,1),UY(J,L,1),
     1              U,V,W,HY)
 90   XY=HX/HY
      X=2*XY-XY**2*ANGLE
      Y=2.6666667*XY-2*XY**3+XY**4*ANGLE
      Z=6.4*XY-2.6666667*XY**3+2*XY**5-XY**6*ANGLE
      IF(IAB.NE.1) GO TO 100
      A=FT(UY(2,L,1),UY(3,L,1),UY(4,L,1),UY(1,L,1))
      B=GT(UY(2,L,1),UY(3,L,1),UY(4,L,1),UY(1,L,1))
      C=HT(UY(2,L,1),UY(3,L,1),UY(4,L,1),UY(1,L,1))
      F=UY(1,L,1)
      GO TO 200
 100  A=FT(UY(NY-1,L,1),UY(NY-2,L,1),UY(NY-3,L,1),
     1              UY(NY,L,1))
      B=GT(UY(NY-1,L,1),UY(NY-2,L,1),UY(NY-3,L,1),
     1              UY(NY,L,1))
      C=HT(UY(NY-1,L,1),UY(NY-2,L,1),UY(NY-3,L,1),
     1              UY(NY,L,1))
 200  SUM=SUM+SIMZ(F,A,B,C,X,Y,Z)
      RETURN
      END
```

```
      SUBROUTINE EVEN(UY,VY, FY,NY,HY,I,K,III,IND,SUM)
C
      REAL*4 UY(NY,2,4),VY(NY,2,4)
      REAL*4 FY(NY,2,2)
C
      FNAA(RO,SO,R,S)=0.5*ALOG((RO-R)**2+(SO-S)**2)
      SIMP(A,B,C,D,E,F,H)=0.3333333333*(A*D+B*E+4*C*F)*H
      FN(A,B,H)=2*H*(ALOG(H)-1)*A+0.1111111111*H**3*
     1          (3*ALOG(H)-1)*B
C
      IF(.NOT.(K.EQ.2.AND.III.EQ.-1)) RETURN
      L=K
      NYY=NY-1
      IF(I/2*2.NE.I) GO TO 20
      I1=I
      I2=I
      SUM=SUM+IND*III*FN(VY(I,K,2),VY(I,K,4),HY)
      GO TO 15
   20 I1=I-1
      I2=I+1
      SUM=SUM+IND*III*FN(VY(I,K,2),VY(I,K,4),HY*2)
   15 DO 10 J=2,NYY,2
      IF(J.EQ.I1.OR.J.EQ.I2) GO TO 10
      X=FNAA(FY(I,K,1),FY(I,K,2),FY(J+1,L,1),FY(J+1,L,2))
      Y=FNAA(FY(I,K,1),FY(I,K,2),FY(J-1,L,1),FY(J-1,L,2))
      Z=FNAA(FY(I,K,1),FY(I,K,2),FY(J,L,1),FY(J,L,2))
      SUM=SUM+IND*III*SIMP(VY(J+1,K,2),VY(J-1,K,2),
     1          VY(J,K,2),X,Y,Z,HY)
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE EEM(UX,VX,M,FX,FY,NX,NY,HX,HY,I,K,L,
     1               KKK,LLL,IND,UY,VY)
      REAL*4 FX(NX,2,2),FY(NY,2,2)
      REAL*4 UX(NX,2,4),VX(NX,2,4)
      REAL*4 UY(NY,2,4),VY(NY,2,4)
      PI=3.141592654
      SUM=0.
      CALL EVEN(UX,VX, FX,NX,HX,I,K,KKK,IND,SUM)
      CALL  CM(UX,VX,FX,FX,NX,NX,HX,I,K,L,LLL,IND,SUM)
      GO TO (10,20,30,40,50),M
   10 CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,K,LLL,IND,SUM)
      CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,L,KKK,IND,SUM)
      GO TO 80
C
   20 CALL  CP(UY,VY,FX,FY,NX,NY,HY,I,K,K,LLL,IND,SUM,
     1               4,NY-1,1,HX)
      CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,L,KKK,IND,SUM)
      GO TO 80
C
   30 CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,K,LLL,IND,SUM)
      CALL  CP(UY,VY,FX,FY,NX,NY,HY,I,K,L,KKK,IND,SUM,
     1               4,NY-1,1,HX)
      GO TO 80
C
   40 CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,K,LLL,IND,SUM)
      CALL  CP(UY,VY,FX,FY,NX,NY,HY,I,K,L,KKK,IND,SUM,
     1               2,NY-3,0,HX)
      GO TO 80
C
   50 CALL  CP(UY,VY,FX,FY,NX,NY,HY,I,K,K,LLL,IND,SUM,
     1               2,NY-3,0,HX)
      CALL  CM(UY,VY,FX,FY,NX,NY,HY,I,K,L,KKK,IND,SUM)
   80 UX(I,K,1)=SUM/PI
      RETURN
      END
```

```
      SUBROUTINE CNA(UX,VX,   FX,FY,NX,NY,HX,HY,UY,VY)
C
      REAL*4 FX(NX,2,2),FY(NY,2,2)
      REAL*4 UY(NY,2,4),VY(NY,2,4)
      REAL*4 UX(NX,2,4),VX(NX,2,4)
C
      PI=3.141592654
      SUM=0.
      CALL  CM(UX,VX,FX,FX,NX,NX,HX,1,1,2,-1,-1,SUM)
      CALL  CM(UY,VY,FX,FY,NX,NY,HY,1,1,2,+1,-1,SUM)
      UX(1,1,1)=2*SUM/PI
      UY(1,1,1)=UX(1,1,1)
      SUM=0.
C
      CALL  CM(UX,VX,FX,FX,NX,NX,HX,NX,1,2,-1,-1,SUM)
      CALL  CM(UY,VY,FX,FY,NX,NY,HY,NX,1,1,-1,-1,SUM)
C
      UX(NX,1,1)=2*SUM/PI
      UY(1,2,1)=UX(NX,1,1)
      RETURN
      END
```

```
      SUBROUTINE CCMUU(UX,UY,VX,VY,FX,FY,NX,NY,HX,HY)
C
      REAL*4 FX(NX,2,2),FY(NY,2,2)
      REAL*4 UX(NX,2,4),UY(NY,2,4)
      REAL*4 VX(NX,2,4),VY(NY,2,4)
C
      NXX=NX-1
      NYY=NY-1
      NNN=MAX0(NXX,NYY)
      DO 100 JJJ=1,3
      I=NX/2
      J=NY/2
C
      DO 70 KK=2,NNN,2
      II=KK/2
      I=I+(-1)**II*(KK-2)
      J=J+(-1)**II*(KK-2)
C
      IF(KK.GT.NXX) GO TO 30
      CALL EEM(VX,UX,1,FX,FY,NX,NY,HX,HY,I,2,1,-1,+1,+1,VY,UY)
      IF(I.EQ.2.OR.I.EQ.NXX) GO TO 10
      CALL EEM(UX,VX,1,FX,FY,NX,NY,HX,HY,I,1,2,+1,-1,-1,UY,VY)
      GO TO 30
   10 IF(I.EQ.NXX) GO TO 20
      CALL EEM(UX,VX,2,FX,FY,NX,NY,HX,HY,I,1,2,+1,-1,-1,UY,VY)
      GO TO 30
   20 CALL EEM(UX,VX,3,FX,FY,NX,NY,HX,HY,I,1,2,+1,-1,-1,UY,VY)
C
C
   30 IF(KK.GT.NYY) GO TO 45
      IF(J.EQ.2.OR.J.EQ.NYY) GO TO 40
      CALL EEM(UY,VY,1,FY,FX,NY,NX,HY,HX,J,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,1,FY,FX,NY,NX,HY,HX,J,2,1,+1,-1,-1,UX,VX)
      GO TO 45
   40 IF(J.EQ.NYY) GO TO 42
      CALL EEM(UY,VY,2,FY,FX,NY,NX,HY,HX,J,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,4,FY,FX,NY,NX,HY,HX,J,2,1,+1,-1,-1,UX,VX)
      GO TO 45
   42 CALL EEM(UY,VY,3,FY,FX,NY,NX,HY,HX,J,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,5,FY,FX,NY,NX,HY,HX,J,2,1,+1,-1,-1,UX,VX)
C
   45 L=I-1
      M=J-1
      IF(KK.GT.NXX.OR.L.EQ.1) GO TO 305
      CALL EEM(VX,UX,1,FX,FY,NX,NY,HX,HY,L,2,1,-1,+1,+1,VY,UY)
      IF(L.EQ.3.OR.L.EQ.NXX-1) GO TO 105
      CALL EEM(UX,VX,1,FX,FY,NX,NY,HX,HY,L,1,2,+1,-1,-1,UY,VY)
      GO TO 305
  105 IF(L.EQ.NXX-1) GO TO 205
      CALL EEM(UX,VX,2,FX,FY,NX,NY,HX,HY,L,1,2,+1,-1,-1,UY,VY)
      GO TO 305
  205 CALL EEM(UX,VX,3,FX,FY,NX,NY,HX,HY,L,1,2,+1,-1,-1,UY,VY)
C
C
  305 IF(KK.GT.NYY.OR.M.EQ.1) GO TO 70
      IF(M.EQ.3.OR.M.EQ.NYY-1) GO TO 405
      CALL EEM(UY,VY,1,FY,FX,NY,NX,HY,HX,M,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,1,FY,FX,NY,NX,HY,HX,M,2,1,+1,-1,-1,UX,VX)
      GO TO 70
  405 IF(M.EQ.NYY-1) GO TO 425
      CALL EEM(UY,VY,2,FY,FX,NY,NX,HY,HX,M,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,4,FY,FX,NY,NX,HY,HX,M,2,1,+1,-1,-1,UX,VX)
      GO TO 70
  425 CALL EEM(UY,VY,3,FY,FX,NY,NX,HY,HX,M,1,2,-1,+1,-1,UX,VX)
      CALL EEM(UY,VY,5,FY,FX,NY,NX,HY,HX,M,2,1,+1,-1,-1,UX,VX)
   70 CONTINUE
  100 CONTINUE
      CALL CNA(UX,VX,   FX,FY,NX,NY,HX,HY,UY,VY)
      RETURN
      END
```

```
      SUBROUTINE FREE(UX,VX,TT,TV,QV,WA,NX,HX,GRAV,TZ,QZ,
     1                MMM,NY)
C
      REAL*4 TZ(NY),QZ(NY)
      REAL*4 UX(NX,2,4),VX(NX,2,4),TT(NX),TV(NX),QV(NX)
      REAL*4 WA(NX)
C
      NXX=NX-1
      KA=4
      KB=KA+1
      Q000=EXP(UX(1,2,1))**3
      Q111=EXP(UX(NX,2,1))**3
      DO 10 I=1,NX
      TT(I)=COS(-VX(I,2,1))*VX(I,2,2)
   10 TV(I)=SIN(-VX(I,2,1))
C
      DO 14 I=2,KA
   14 CALL TRAPE(TV,TT,I,HX,WA(I))
      DO 18 I=KB,NX
   18 CALL ZTRAP(TV,I,HX,WA(I),QV)
C
      GRAV=(Q000-Q111)/(3*WA(NX))
      DO 20 I=2,NXX
   20 UX(I,2,1)=ALOG(Q000-3*GRAV*WA(I))/3
      CALL XTAN(UX,NX,HX,2,+1,+1,GRAV,TZ,QZ,MMM,NY)
      RETURN
      END
```

```
      SUBROUTINE TRAPB(F,FT,N,H,APPR)
      REAL*4 F(N),FT(N)
      APPR=0.5*(F(1)-F(N))
      DO 10 I=2,N
   10 APPR=APPR+F(I)
      APPR=APPR*H
      APPR=APPR+0.0833333333333*H*H*(FT(1)-FT(N))
      RETURN
      END
```

```
      SUBROUTINE ZTRAP(Z,N,H,APPR,F)
      REAL*4 Z(N)
      REAL*4 F(N)
      REAL*4 COEF(6)
      DO 1 I=1,N
    1 F(I)=Z(I)
      APPR=0.5*(F(1)-F(N))
      DO 10 I=2,N
   10 APPR=APPR+F(I)
      APPR=APPR*H
      M=N
      COEF(1)=0.083333333
      COEF(2)=0.041666666655
      COEF(3)=0.02638888888
      COEF(4)=0.01875
      COEF(5)=0.0142691799
      COEF(6)=0.0113673942
      DO 20 J=1,6
      IF(M.EQ.2) GO TO 40
      DO 30 I=2,M
   30 F(I-1)=F(I)-F(I-1)
      M=M-1
   20 APPR=APPR-H*COEF(J)*(F(M)+(-1)**J*F(1))
   40 RETURN
      END
```

```
      SUBROUTINE PRINTG(UV,JA,JB,MMM,KX,KY,MM)
      REAL*4 UV(KX,KY)
      INTEGER MM(KX)
      DO 10 I=1,KX
   10 MM(I)=MMM+I-KX
      PRINT20,(MM(I),I=1,KX)
   20 FORMAT(5X,6I11)
      PRINT,' '
      DO 30 J=1,KY
      PRINT50,J,(UV(I,J),I=1,KX)
   50 FORMAT(1X,I3,1X,6F11.6)
      IF(J/5*5.NE.J) GO TO 30
      PRINT,' '
      IF(J/10*10.NE.J) GO TO 30
      PRINT,' '
   30 CONTINUE
      PRINT99,MMM
   99 FORMAT('1',I4)
      RETURN
      END
```

```
      SUBROUTINE PARA(UX,VX,TT,TV,QV,X,Y,YA,NX,HX,FF,
     1                GRAV,Q,COSV,SINV,UY,NY,HY,TZ)
C
      REAL*4 TZ(NY)
      REAL*4 UY(NY,2,4)
      REAL*4 Q(NX),COSV(NX),SINV(NX)
      REAL*4 UX(NX,2,4),VX(NX,2,4)
      REAL*4 TT(NX),TV(NX),QV(NX),X(NX),Y(NX),YA(NX)
C
      KA=4
      KB=KA+1
C
      QDD=EXP(UX(1,2,1))**2
      Y(1)=0.
      DO 10 I=2,NX
   10 Y(I)=0.5/GRAV*(QDD-EXP(UX(I,2,1))**2)
      AH=Y(NX)
C
      X(1)=0.
      DO 20 I=1,NX
      COSV(I)=COS(-VX(I,2,1))
      SINV(I)=SIN(-VX(I,2,1))
      Q(I)=EXP(UX(I,2,1))
      TV(I)=COSV(I)/Q(I)
   20 TT(I)=-SINV(I)/Q(I)*VX(I,2,2)-COSV(I)/Q(I)*UX(I,2,2)
C
      DO 22 I=2,KA
   22 CALL TRAPE(TV,TT,I,HX,X(I))
      DO 24 I=KB,NX
   24 CALL ZTRAP(TV,I,HX,X(I),QV)
C
      WL=X(NX)
C
C
      YA(1)=0.
      DO 30 I=1,NX
      TV(I)=SINV(I)/Q(I)
   30 TT(I)=COSV(I)/Q(I)*VX(I,2,2)-SINV(I)/Q(I)*UX(I,2,2)
      DO 32 I=2,KA
   32 CALL TRAPE(TV,TT,I,HX,YA(I))
      DO 34 I=KB,NX
   34 CALL ZTRAP(TV,I,HX,YA(I),QV)
C
      C=FF/WL
C
      DO 50 I=1,NX
   50 TV(I)=1./EXP(UX(I,2,1))**2
      CALL ZTRAP(TV,NX,HX,CWW,QV)
      TIME=CWW
      UC=C-WL/TIME
C
      DO 60 I=1,NX
   60 TV(I)=EXP(UX(I,1,1))
      CALL ZTRAP(TV,NX,HX,C1,QV)
      C1=C1/WL
      D=0.5/GRAV*(QDD-C1)
```

```
C
      DO 70 I=1,NY
   70 TZ(I)=1./EXP(UY(I,1,1))
      CALL ZTRAP(TZ,NY,HY,H1,TZ)
C
      H=H1+D
      WL=2*WL
      TIME=TIME*2
      HL=H/WL
      CH=C*C/GRAV/H
      AHH=AH/H
C
      PRINT,' '
      PRINT105,AH,D,H1,H
      PRINT,' '
      PRINT106,TIME,GRAV,WL
      PRINT,' '
      PRINT107,C,C1,U0
      PRINT,' '
      PRINT108,HL,CH,AHH
  105 FORMAT(3X,'A= ',F10.5,5X,'D= ',F10.5,5X,'H1=',F10.5,
     1                 5X,'H= ',F10.5)
  106 FORMAT(3X,'T1=',F10.5,3X,'GRAV=',F10.5,5X,'WL=',
     1                 F10.5)
  107 FORMAT(3X,'C= ',F10.5,3X,'C1C1=',F10.5,5X,'U0=',
     1                 F10.5)
  108 FORMAT(1X,'H/WL=',F10.5,2X,'CC/GH=',F10.5,4X,'A/H=',
     1                 F10.5)
      AL=AH/WL
      PRINT,' '
      PRINT109,AL
  109 FORMAT(1X,'A/WL=',F10.5)
C
      PRINT,' '
      PRINT,' '
      PRINT,' '
      PRINT,' '
      PRINT88
   88 FORMAT(6X,'I',3X,'VELOCITY',6X,'ANGLE',10X,'X',12X,
     1                 'Y',12X,'YA')
      PRINT,' '
      DO 90 I=1,NX
   90 PRINT100,I,EXP(UX(I,2,1)),-VX(I,2,1)*180/3.141592654,
     1                 X(I),Y(I),YA(I)
XTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
XTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
  100 FORMAT(1X,I4,5F13.8)
      PRINT99,I
   99 FORMAT('1',I3)
      RETURN
      END

$ENTRY
```

APPENDIX C

```
$JOB     WATFIV  XXXXXXXXXX T.H.LIM
C
C         2-D VERTICAL JET FROM A LARGE VESSEL
C         USING CAUCHY INTEGRAL METHOD
C
C
C
          INTEGER MMMM(6)
          REAL*4 UX(69,2,3),VX(69,2,3),UV(6,227)
          REAL*4 WA(25),UV(25),TV(25),P(3),V(2)
          REAL*4 X(69),Y(69),TT(25)
C
C
C         NX=NUMBER OF POINTS ON HORIZONTAL SIDES
C         HX=LENGTH OF SUBINTERVALS ON HORIZONTAL SIDES
C         UX=U ALONG HORIZONTAL SIDES
C         VX=V ALONG HORIZONTAL SIDES
C
C
C
          IA=25
          IB=45
          NX=69
C
          Q0=0.008
          XX=4.65
          SMALL=0.03
          Q1=1.15
          QC=1.0
          YY=1.
          HX=XX/(NX-1)
          NXX=NX-1
          IC=5
          IL=NX+1-IB
          Q13=Q1*3
          QC3=QC**3
          PI=3.14159254
C
C
C
C
C         SUBROUTINE XTAN---COMPUTE DERIVATIVES OF U OR V
C                           ON HORIZONTAL SIDES
C
C         SUBROUTINES EMAA,EMBB AND EMCC---COMPUTE 3 INTEGRALS
C                                 IN CAUCHY INTEGRAL EQUATIONS
C
C         SUBROUTINE EEM---COMPUTE U OR V USING EMAA,EMBB
C                           AND EMCC
C
C         SUBROUTINE COMUU---COMPUTE U OR V  USING EEM
C                             ON HORIZONTAL SIDES
C
C         SUBROUTINE FREE---COMPUTE U ON FREE SURFACE USING
C                           FREE SURFACE CONDITION
C         SUBROUTINES TRAPB AND ZTRAP --- ARE IMPROVED
C                                   TRAPEZOIDAL RULES
C
C         SUBROUTINE PRINTS---PRINT U AND V
C         SUBROUTINE PARA---COMPUTE THE PARAMETERS
C
```

```
      DO 12 I=1,NX
      DO 12 J=1,2
      DO 12 K=1,3
      UX(I,J,K)=0.
   12 VX(I,J,K)=0.
C
      VX(1,1,1)=Q0
      VX(1,2,1)=Q0
      VX(NX,1,1)=Q1
      VX(NX,2,1)=Q1
C
      FFA=0.4
      VX(IB+1,1,1)=0.157*PI*FFA
      VX(IB+2,1,1)=0.25*PI*FFA
      VX(IB+3,1,1)=0.31*PI*FFA
      VX(IB+4,1,1)=0.345*PI*FFA
C
      II=IB+5


      JJ=NX-3
      ISUM=0
      DO 5 I=II,JJ
    5 ISUM=ISUM+JJ+1-I
      TEMP=(0.5-0.345*FFA)*PI/ISUM
      DO 6 I=II,JJ
    6 VX(I,1,1)=VX(I-1,1,1)+(JJ+1-I)*TEMP
      DO 7 I=1,3
    7 VX(NX+1-I,1,1)=0.5*PI
C
      JJ=NX
      TEMP=(Q1-Q0)/(JJ-IB)
      DO 10 I=IB,JJ
      T=Q0+(I-IB)*TEMP
      UX(I,1,1)=T*COS(VX(I,1,1))
   10 VX(I,1,1)=T*SIN(VX(I,1,1))
C
      ISUM=0
      DO 20 I=2,JJ
   20 ISUM=ISUM+I-1
      TEMP=(Q1-Q0)/ISUM
      DO 25 I=2,JJ
   25 VX(I,2,1)=VX(I-1,2,1)+(I-1)*TEMP
C
      ISUM=0
      II=IA+1
      DO 30 I=II,IB
   30 ISUM=ISUM+I-IA
      TEMP=Q0/ISUM
      DO 35 I=II,IB
   35 UX(I,1,1)=UX(I-1,1,1)+(I-IA)*TEMP
C
      ISUM=0
      IT=IA/2+1
      FACTOR=5.
      DO 39 I=2,IT
   39 ISUM=ISUM+I-1
      TEMP=Q0/ISUM*FACTOR
      DO 40 I=2,IT
      J=IA+I-1
      VX(J,1,1)=VX(J+1,1,1)+(I-1)*TEMP*(FACTOR+1)/FACTOR
   40 VX(I,1,1)=VX(I-1,1,1)+(I-1)*TEMP
```

```
C
      CALL FREE(UX,VX,P,  TV,QV,WA,NX,HX,IA,IB,IL,GRAV,
     1            QC3,Q13,V)
      CALL PARA(UX,VX,NX,HX,X,Y,WA,TV,QV,TT,IL,GRAV)
      CALL XTAN(UX,VX,P,UX,HX,2,IA,IB,V)
      N1=2*NX
      MM=0
      PRINT99,MM
   99 FORMAT('1',I4)
      DO 38 I=1,IC
      UV(I,NX+IA)=0.
   38 UV(I,N1+IA)=0.
C
      XX=XX-SMALL
   50 XX=XX+SMALL
      HX=XX/(NX-1)
      DO 56 MMM=1,30
      MM=MM+1
      DO 5700 I=1,NX
      UV(MM,I)=VX(I,2,1)
      IF(I.EQ.IA) GO TO 5700
      UV(MM,NX+I)=(UX(I,1,1)**2+VX(I,1,1)**2)**0.5
      UV(MM,N1+I)=ATAN2(-VX(I,1,1),UX(I,1,1))*180/PI
 5700 CONTINUE
      IF(MMM.GT.8) GO TO 582
      DO 580 I=1,NX
  580 PRINT581,I,UV(MM,I),UV(MM,NX+I),UV(MM,N1+I)
  581 FORMAT(5X,I10,3F25.10)
  582 CONTINUE
      IF(MM.NE.IC) GO TO 58
      CALL PRINTG(UV,1,10,MMM,IC,3*NX,MMMM)
      MM=0
   58 CONTINUE
C
      CALL COMUU(UX,VX, P,NX,HX,IA,IB,V)
      CALL FREE(UX,VX,P,  TV,QV,WA,NX,HX,IA,IB,IL,GRAV,
     1            QC3,Q13,V)
C
   56 CONTINUE
      CALL PARA(UX,VX,NX,HX,X,Y,WA,TV,QV,TT,IL,GRAV)
      MM=0
      GO TO 50
      STOP
**WARNING**  UNNUMBERED EXECUTABLE STATEMENT FOLLOWS A TRANSFER
      END
```

```
      SUBROUTINE XTAN(W,U,P,NX,HX,K,IA,IB,V)
      REAL*4 U(NX,2,3),W(NX,2,3),P(3),V(2)
      FT(A,B,C,D,E,F)=(45*(A-B)-9*(C-D)+(E-F))/(60*HX)
      GT(A,B,C,D,E,F,G)=(-490*A+270*(B+C)-27*(D+E)+
     1                  2*(F+G))/(180*HX*HX)
      HT(A,B,C,D,E,F)=(-13*(A-B)+8*(C-D)-(E-F))/(8*HX**3)
      FA(A,B,C,D,E,F,G)=-(-147*A+360*B-450*C+400*D-225*E
     1                  +72*F-10*G)/(60*HX)
      HA(A,B,C,D,E,F,G)=-(-49*A+232*B-461*C+496*D-307*E
     1                  +104*F-15*G)/(8*HX**3)
      FB(A,B,C,D,E,F,G)=-(-10*A-77*B+150*C-100*D+50*E-
     1                  15*F+2*G)/(60*HX)
      HB(A,B,C,D,E,F,G)=-(-15*A+56*B-83*C+64*D-29*E+8*F-G)
     1                  /(8*HX**3)
      FC(A,B,C,D,E,F,G)=-(2*A-24*B-35*C+80*D-30*E+8*F-G)/
     1                  (60*HX)
      HC(A,B,C,D,E,F,G)=-(-A-8*B+35*C-48*D+29*E-8*F+G)/
     1                  (8*HX**3)
C
      KK=NX-3
C
      U(1,K,2)=-FA(U(1,K,1),U(2,K,1),U(3,K,1),U(4,K,1),
     1                  U(5,K,1),U(6,K,1),U(7,K,1))
      U(2,K,2)=-FB(U(1,K,1),U(2,K,1),U(3,K,1),U(4,K,1),
     1                  U(5,K,1),U(6,K,1),U(7,K,1))
      U(3,K,2)=-FC(U(1,K,1),U(2,K,1),U(3,K,1),U(4,K,1),
     1                  U(5,K,1),U(6,K,1),U(7,K,1))
C
      J=NX
      U(J,K,2)=FT(P(1),U(J-1,K,1),P(2),U(J-2,K,1),P(3),
     1                  U(J-3,K,1))
      U(J,K,3)=HT(P(1),U(J-1,K,1),P(2),U(J-2,K,1),P(3),
     1                  U(J-3,K,1))
      J=NX-1
      U(J,K,2)=FT(U(J+1,K,1),U(J-1,K,1),P(1),U(J-2,K,1),
     1                  P(2),U(J-3,K,1))
      U(J,K,3)=HT(U(J+1,K,1),U(J-1,K,1),P(1),U(J-2,K,1),
     1                  P(2),U(J-3,K,1))
      J=NX-2
      U(J,K,2)=FT(U(J+1,K,1),U(J-1,K,1),U(J+2,K,1),
     1                  U(J-2,K,1),P(1),U(J-3,K,1))
      U(J,K,3)=HT(U(J+1,K,1),U(J-1,K,1),U(J+2,K,1),
     1                  U(J-2,K,1),P(1),U(J-3,K,1))
C
      IF(K.EQ.1) GO TO 20
      DO 10 I=4,KK
   10 U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1                  U(I-2,K,1),U(I+3,K,1),U(I-3,K,1))
      RETURN
C
   20 II=IA-3
      JJ=IA+3
      I=IA
      J=IA
      U(J,K,2)=FA(U(I,K,1),U(I-1,K,1),U(I-2,K,1),
     1   U(I-3,K,1),U(I-4,K,1),
     1   U(I-5,K,1),U(I-6,K,1))
      J=IA-1
      U(J,K,2)=FB(U(I,K,1),U(I-1,K,1),U(I-2,K,1),
     1   U(I-3,K,1),U(I-4,K,1),
     1   U(I-5,K,1),U(I-6,K,1))
      J=IA-2
      U(J,K,2)=FC(U(I,K,1),U(I-1,K,1),U(I-2,K,1),
     1   U(I-3,K,1),U(I-4,K,1),
     1   U(I-5,K,1),U(I-6,K,1))
      DO 25 I=4,II
   25 U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1                  U(I-2,K,1),U(I+3,K,1),U(I-3,K,1))
```

```
      J=IA
      W(J,K,2)=FT(W(J+1,K,1),-W(J+1,K,1),W(J+2,K,1),
     1    -W(J+2,K,1),
     1       W(J+3,K,1),-W(J+3,K,1))
      J=IA+1
      W(J,K,2)=FT(W(J+1,K,1),W(J-1,K,1),W(J+2,K,1),
     1    -W(J,K,1),
     1       W(J+3,K,1),-W(J+1,K,1))
      J=IA+2
      W(J,K,2)=FT(W(J+1,K,1),W(J-1,K,1),W(J+2,K,1),
     1    W(J-2,K,1),
     1       W(J+3,K,1),-W(J-1,K,1))
      DO 30 I=IB,KK
      U(I,K,2)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1    U(I-2,K,1),
     1    U(I+3,K,1),U(I-3,K,1))
   30 U(I,K,3)=FT(U(I+1,K,1),U(I-1,K,1),U(I+2,K,1),
     1    U(I-2,K,1),
     1    U(I+3,K,1),U(I-3,K,1))
C
      DO 60 I=JJ,KK
   60 W(I,K,2)=FT(W(I+1,K,1),W(I-1,K,1),W(I+2,K,1),
     1    W(I-2,K,1),
     1       W(I+3,K,1),W(I-3,K,1))
C
      J=NX-1
      W(J,K,2)=FT(W(J+1,K,1),W(J-1,K,1),0.,W(J-2,K,1),0.,
     1    W(J-3,K,1))
      J=NX-2
      W(J,K,2)=FT(W(J+1,K,1),W(J-1,K,1),W(J+2,K,1),
     1       W(J-2,K,1),
     1       0.,W(J-3,K,1))
C
      I=IB
      V(1)=GT(U(I,1,1),U(I+1,1,1),U(I-1,1,1),U(I+2,1,1),
     1    U(I-2,1,1),
     1       U(I+3,1,1),U(I-3,1,1))
      I=IB+1
      V(2)=GT(U(I,1,1),U(I+1,1,1),U(I-1,1,1),U(I+2,1,1),
     1    U(I-2,1,1),
     1       U(I+3,1,1),U(I-3,1,1))
      RETURN
      END
```

```
      SUBROUTINE EMAA(U,NX,HX,I,K,IX,IY,SUM)
      REAL*4 U(NX,2,3)
      FNAA(J)=ALOG(1.+((I-J)*HX)**2)
      SUMB=0.
      SUMC=0.
      SUMA=U(IX-1,K,2)*FNAA(IX-1)-U(IY+1,K,2)*FNAA(IY+1)
      DO 20 J=IX,IY,2
      SUMB=SUMB+U(J,K,2)*FNAA(J)
   20 SUMC=SUMC+U(J+1,K,2)*FNAA(J+1)
      SUM=SUM+0.5*(SUMA+4.*SUMB+2.*SUMC)
      RETURN
      END

      SUBROUTINE EMCC(U,NX,HX,I,K,IX,IY,SUM)
      REAL*4 U(NX,2,3)
      FNAA(J)=ALOG(IABS(I-J)*HX)
      SUMB=0.
      SUMC=0.
      SUMA=U(IX-1,K,2)*FNAA(IX-1)-U(IY+1,K,2)*FNAA(IY+1)
      DO 20 J=IX,IY,2
      SUMB=SUMB+U(J,K,2)*FNAA(J)
   20 SUMC=SUMC+U(J+1,K,2)*FNAA(J+1)
      SUM=SUM+SUMA+4.*SUMB+2.*SUMC
      RETURN
      END

      SUBROUTINE EMBB(V,NX,HX,I,K,IX,IY,SUM)
      REAL*4 V(NX,2,3)
      FNXX(J)=1./(1.+((I-J)*HX)**2)
      SUMB=0.
      SUMC=0.
      SUMA=V(IX-1,K,1)*FNXX(IX-1)-V(IY+1,K,1)*FNXX(IY+1)
      DO 20 J=IX,IY,2
      SUMB=SUMB+V(J,K,1)*FNXX(J)
   20 SUMC=SUMC+V(J+1,K,1)*FNXX(J+1)
      SUM=SUM+SUMA+4.*SUMB+2.*SUMC
      RETURN
      END
```

```
      SUBROUTINE EIM(UX,VX,P,NX,HX,I,IA,IB,V)
      REAL*4 UX(NX,2,3),VX(NX,2,3),P(3),V(2)
      FN(A,B,H)=2*H*(ALOG(H)-1)*A+0.1111111111*H**3*(3*ALOG(H)-1)*B
      GN(A,B,C,D)=0.5*HX*(A+B)+0.0833333333*HX*HX*(C-D)
      HN(A,B,C,D)=0.125*(A-5*B+19*C+9*D)
      FNAA(J)=ALOG(IABS(I-J)*HX)
      FNRR(J)=1./((J-I)*HX)
      PI=3.141592654
      NXX=NX-1
      IAX=IA-1
      IAY=IA+1
      IBX=IB-1
      IBY=IB+1
      AC=VX(1,1,1)*ATAN2(1.,(I-1)*HX)+VX(NX,1,1)*
     1               ATAN2(1.,(NX-I)*HX)
      SUM=0.
      CALL EMAA(UX,NX,HX,I,1,IAY,NXX,SUM)
      CALL EMBB(VX,NX,HX,I,1,2,IAX,SUM)
      CALL EMBB(VX,NX,HX,I,1,IBY,NXX,SUM)
      VX(I,2,1)=(AC+0.3333333333*HX*SUM)/PI
C
      IF(I.LT.IA) GO TO 100
      IF(I.GT.IB) GO TO 200
      IF(I.EQ.IA.OR.I.EQ.IB) RETURN
C
  300 SUM=0.
      CALL EMCC(VX,NX,HX,I,1,2,IAX,SUM)
      CALL EMCC(VX,NX,HX,I,1,IBY,NXX,SUM)
      SUM=-SUM
      CALL EMAA(VX,NX,HX,I,2,2,NXX,SUM)
      UX(I,1,1)=   0.3333333333*HX*SUM/PI
      RETURN
C
  100 SUM=0.
      CALL EMCC(UX,NX,HX,I,1,IAY,NXX,SUM)
      CALL EMBB(VX,NX,HX,I,2,2,NXX,SUM)
      VX(I,1,1)=(AC+0.3333333333*HX*SUM)/PI
      RETURN
C
  200 SUMA=-FN(VX(I,1,2),VX(I,1,3),HX)
      SUM=0.
      IF(I/2*2.NE.I) GO TO 220
      I1=I-2
      I2=I+2
      GO TO 240
  220 I1=I-3
      I2=I+3
      SUM=SUM+HN(VX(I+4,1,2)*FNAA(I+4),VX(I+3,1,2)*
     1               FNAA(I+3),
     1     VX(I+2,1,2)*FNAA(I+2),VX(I+1,1,2)*FNAA(I+1))
      IF(I.GT.IB+2) GO TO 230
      SUMA=SUMA-GN(VX(I-2,1,2)*FNAA(I-2),VX(I-1,1,2)*
     1               FNAA(I-1),
     1               VX(I-2,1,2)*FNRR(I-2)+V(1)*FNAA(I-2),
     1               VX(I-1,1,2)*FNRR(I-1)+V(2)*FNAA(I-1))
      GO TO 240
  230 SUM=SUM+HN(VX(I-4,1,2)*FNAA(I-4),VX(I-3,1,2)*
     1               FNAA(I-3),
     1     VX(I-2,1,2)*FNAA(I-2),VX(I-1,1,2)*FNAA(I-1))
  240 CALL EMCC(VX,NX,HX,I,1,2,IAX,SUM)
      IF(I1.LT.IB) GO TO 260
      CALL EMCC(VX,NX,HX,I,1,IBY,I1,SUM)
  260 CALL EMCC(VX,NX,HX,I,1,I2,NXX,SUM)
      SUM=-SUM
      CALL EMAA(VX,NX,HX,I,2,2,NXX,SUM)
      UX(I,1,1)=(SUMA+0.3333333333*HX*SUM)/PI
      RETURN
      END
```

```
      SUBROUTINE CCMUU(UX,VX,P,NX,HX,IA,IB,V)
      REAL*4 UX(NX,2,3),VX(NX,2,3),P(3),V(2)
      NXX=NX-1
      DO 100 JJJ=1,3
      I=NX/2
      DO 70 KK=2,NXX,2
      II=KK/2
      I=I+(-1)**II*(KK-2)
      IF(I.LT.2.OR.I.GT.NX-4) GO TO 30
      CALL BEM(UX,VX,P, NX,HX,I,IA,IB,V)
   30 L=I-1
      IF(L.LT.2.OR.L.GT.NX-4) GO TO 70
      CALL BEM(UX,VX,P, NX,HX,L,IA,IB,V)
   70 CONTINUE
  100 CONTINUE
       CALL XTAN(UX,VX,P,NX,HX,2,IA,IB,V)
      RETURN
      END


      SUBROUTINE FREE(UX,VX,P,  TV,QV,WA,NX,HX,IA,IB,IC,
     1             GRAV,QC3,Q13,V)
      REAL*4 UX(NX,2,3),VX(NX,2,3),P(3), TV(IC),QV(IC)
      REAL*4 WA(IC),V(2)
      DO 100 I=IB,NX
      J=NX+1-I
  100 TV(J)=-SIN(ATAN2(VX(I,1,1),UX(I,1,1)))
      DO 200 I=2,IC
  200 CALL ZTRAP(TV,I,HX,WA(I),QV)
      GRAV=(QC3-Q13)/(3.*WA(IC))
      PRINT,'GRAV=',GRAV
      DO 300 I=2,IC
  300 VX(NX+1-I,1,1)=-TV(I)*(Q13+3.*GRAV*WA(I))**
     1             0.333333333
      DO 310 I=1,3
      VX(NX-I,2,1)=VX(NX-I,1,1)
  310 P(I)=(Q13+3.*GRAV*I*HX)**0.333333333
      PRINT,'P(1),P(2),P(3) = ',P(1),P(2),P(3)
       CALL XTAN(UX,VX,P,NX,HX,1,IA,IB,V)
      RETURN
      END
```

```
      SUBROUTINE ZTRAP(Z,N,H,APPR,F)
      REAL*4 Z(N),F(N),COEF(6)
      DO 1 I=1,N
1     F(I)=Z(I)
      APPR=0.5*(F(1)-F(N))
      DO 10 I=2,N
10    APPR=APPR+F(I)
      APPR=APPR*H
      M=N
      COEF(1)=0.083333333
      COEF(2)=0.041666666666
      COEF(3)=0.026388888884
      COEF(4)=0.01875
      COEF(5)=0.014269179
      COEF(6)=0.0113673942
      DO 20 J=1,6
      IF(M.EQ.2) GO TO 40
      DO 30 I=2,M
30    F(I-1)=F(I)-F(I-1)
      M=M-1
20    APPR=APPR-H*COEF(J)*(F(M)+(-1)**J*F(1))
40    RETURN
      END
```

```
      SUBROUTINE PRINTG(UV,JA,JB,MMM,KX,KY,MM)
      REAL*4 UV(KX,KY)
      INTEGER MM(KX)
      DO 10 I=1,KX
10    MM(I)=MMM+I-KX
      PRINT20,(MM(I),I=1,KX)
20    FORMAT(5X,6I11)
      PRINT,' '
      DO 30 J=1,KY
      PRINT50,J,(UV(I,J),I=1,KX)
50    FORMAT(1X,I3,1X,6F11.6)
      IF(J/5*5.NE.J) GO TO 30
      PRINT,' '
      IF(J/10*10.NE.J) GO TO 30
      PRINT,' '
30    CONTINUE
      PRINT 99,MMM
99    FORMAT('1',I4)
      RETURN
      END
```

```
      SUBROUTINE PARA(UX,VX,NX,HX,X,Y,WA,TV,QV,TT,IL,GRAV)
      REAL*4 UX(NX,2,3),VX(NX,2,3),X(NX),Y(NX)
      REAL*4 WA(IL),TV(IL),QV(IL),TT(IL)
      DO 100 I=1,NX
  100 X(NX+1-I)=1./VX(I,2,1)
      CALL ZTRAP(X,NX,HX,H0,Y)
      PRINT,' '
      PRINT,'H0= ',H0
      PRINT,' '
      IB=NX+1-IL
      DO 200 I=IB,NX
      J=NX+1-I
      X(J)=(UX(I,1,1)**2+VX(I,1,1)**2)**0.5
      Y(J)=ATAN2(-VX(I,1,1),UX(I,1,1))
      TV(J)=COS(Y(J))/X(J)
      QV(J)=SIN(Y(J))/X(J)
  200 Y(J)=Y(J)*180/3.14159265
      DO 300 I=2,IL
  300 CALL ZTRAP(TV,I ,HX,WA(I),TT)
      DO 400 I=2,IL
  400 CALL ZTRAP(QV,I ,HX,TV(I),TT)
      DO 500 I=2,IL
  500 QV(I)=(1.-X(I)**2)/GRAV*0.5
      WA(1)=0.
      TV(1)=0.
      QV(1)=0.
      PRINT520
  520 FORMAT('1', 9X,'I',16X,'X(I)',18X,'Y(I)')
      PRINT,' '
      DO 550 J=1,IL
      I=J
      PRINT,' '
  550 PRINT600,I,WA(IL)-WA(IL+1-I),TV(IL)-TV(IL+1-I)
*EXTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
*EXTENSION*  OTHER COMPILERS MAY NOT ALLOW EXPRESSIONS IN OUTPUT LISTS
  600 FORMAT(1X,I10,2F22.7)
      PRINT520
      RETURN
      END

  $ENTRY
```

APPENDIX D

```
C     2-D VERTICAL JET FROM AN INFINITE CHANNEL
C     USING A CONFORMAL MAPPING AND RIEMANN-HILBERT
C          SOLUTION TO A MIXED-BOUNDARY-VALUE PROBLEM
C     WRITTEN BY T.H.LIM , DEC.28,1977.
C
      REAL*4 X(21),Y(21),T(21),TT(21)
      REAL*4 ZZ(21),Z(21,2,3),GS(21)
      REAL*4 G(21),B(21),CS(21),SN(21),GZ(21,21)
      F2=1.2
      QE=1.2
      K12=5
      N=21
      ER=0.025
      TD=0.0005
      C2=2.5
      FACTOR=3.
      TB=4.5
C
C
C
C     N= NUMBER OF POINTS
C     ER=LENGTH OF HALF-INTERVAL AROUND SINGULARITY
C     QE=NORMALIZED VELOCITY AT THE POINT E
C     TB=ONE POINT ON IM(T)=0.
C     CP=LOCAL UPSTREAM PRESSURE COEFFICIENT
C     H=STEP SIZE OF SUB-INTERVAL
C     (X,Y)=NORMALIZED COORDINATES OF FREE SURFACE ED
C     G =INTEGRAND OF I(1)
C     GS=INTEGRAND OF I(2)
C     T=MESH POINTS ON THE INTERVAL (TD,1)
C     TT=A FACTOR OF INTEGAL J
C     RF=2/F**2
C     B=A FACTOR OF INTEGRAL J
C     CS=INTEGRAND FOR COMPUTING X
C     SN=INTEGRAND FOR COMPUTING Y
C
C
C     SIMF=ROUTINE FOR COMPUTING INTEGRAL I(1)
C     SIMG=ROUTINE FOR COMPUTING INTEGRAL I(2)
C     SIMFX=ROUTINE FOR COMPUTING X AND Y
C     CUHI=ROUTINE FOR LAGRANGIAN CUHI-INTERPOLATION
C
      S=1./C2
      CP=QE*QE-1.
      PI=3.141592654
      M=N/2-1
      NN=N-1
      R=0.5
      H=(1.-TD)/NN
      GS(1)=0.
      GS(N)=0.
      X(1)=0.
```

```
      ZNN=0.
      DO 5 I=1,NN
    5 ZNN=ZNN+I**FACTOR
      SMALL=R/ZNN
      Y(1)=0.
      DO 6 I=1,NN
    6 Y(I+1)=Y(I)-I*SMALL**FACTOR
      Y(N)=4.*(-1)
C
      ERR=ER*H
      H3=(H-ERR)/3.
      H3H=H3/H
C
      DO 10 I=1,N
      T(I)=-1+(I-1)*H
      TT(I)=SQRT(ABS((1.+T(I))*T(I)))
   10 CONTINUE
      RF=2./F2
C
      DO 1000 MMM=1,20
      DO 100 I=1,N
      B(I)=0.5*ARSIN(1+2*T(I)+2*T(I)*(T(I)+1.)/
     1         (TB-T(I)))+0.25*PI
  100 CONTINUE
      DO 150 I=2,NN
  150 G(I)=ALOG(QE*QE-RF*Y(I))/TT(I)
C
      DO 200 I=2,NN
      DO 200 J=2,NN
      IF(I.EQ.J) GO TO 200
      GZ(I,J)=G(J)/((J-I)*H)
  200 CONTINUE
C
      CALL CUBI(G,Z,ZZ,N,NN,H,H3,H3H,ERR)
      CALL SIMP(G,Z,N,NN,H,H3,TGD)
C
      DO 300 I=2,NN
      CALL SIMPG(GZ,N,NN,H,I,GS(I),ZZ)
  300 GS(I)=-0.5/PI*TT(I)*GS(I)
C
      DO 400 I=1,N
      BGS=B(I)+GS(I)
      TEMP=T(I)*SQRT(QE*QE-RF *Y(I))
      CS(I)=-COS(BGS)/TEMP
  400 SN(I)= SIN(BGS)/TEMP
C
      DO 450 I=2,N
      CALL SIMPX(CS,N,I,H,X(I))
      CALL SIMPX(SN,N,I,H,Y(I))
  450 CONTINUE
      R=-Y(N)
      BY=S+X(N)
      CC=S/BY
```

```
          TB=0.25*(1-EXP(TGD))**2/EXP(TGD)
C
          PRINT,' '
          PRINT,MMM
C
          IF (MMM.GE.7) GO TO 1001
          PRINT,' '
          DO 600 I=1,N
  600     PRINT610,I,X(I),Y(I)
  610     FORMAT(1X,I10,2F20.10)
          PRINT,' '
          PRINT,' '
          GO TO 1000
C
 1001     PRINT1002,MMM
 1002     FORMAT('1',I10)
          DO 1003 I=1,10
 1003     PRINT,' '
C
          PRINT1004,QE
          PRINT,' '
          PRINT1005,CP
          PRINT,' '
          PRINT1006,F2
          PRINT,' '
          PRINT,' '
 1004     FORMAT(13X,'QE=',F10.7)
 1005     FORMAT(13X,'CP=',F10.7)
 1006     FORMAT(13X,'F2=',F10.7)
C
          PRINT1007
 1007     FORMAT(18X,'I',16X,'X(I)',18X,'Y(I)')
          PRINT,' '
          DO 1008 I=1,N
          PRINT,' '
 1008     PRINT1009,I,X(I),Y(I)
 1009     FORMAT(1X,I18,2F22.7)
          DO 1010 I=1,4
 1010     PRINT,' '
          PRINT1011,K12
 1011     FORMAT(30X,'TABLE 6.'I2)
          PRINT1002
C
C
 1000     CONTINUE
          STOP
          END
```

```
      SUBROUTINE CUBI(G,Z,ZZ,N,II,H,H3,H3H,ERR)
      REAL*4 Z( N,2,3),ZZ(N),G(N)
      Y(A,B,C,D,X)=-(X-2)*(X-3)*(X-4)/6*A+
     1     0.5*(X-1)*(X-3)*(X-4)*B-
     1     0.5*(X-1)*(X-2)*(X-4)*C+(X-1)*(X-2)*(X-3)*D/6
      SIMP(A,B,C)=H3*0.3333333333*(A+4*B+C)
      QUAD(A,B,C,D)=H3*0.375*(A+3*B+3*C+D)
C
      H3H2=H3H*2
      H3H3=H3H*3
      ERR1=ERR+H3
      ERR2=ERR+H3*2
      H32=H3*2
      H33=H3*3
C
      JJ=N-3
      DO 100 I=4,JJ
      Z(I,1,1)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H)
      Z(I,1,2)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H2)
      Z(I,1,3)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H3)
      Z(I,2,1)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H)
      Z(I,2,2)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H2)
      Z(I,2,3)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H3)
  100 CONTINUE
C
      I=2
      Z(I,1,1)=Y(G(I),G(I+1),G(I+2),G(I+3),H3H)
      Z(I,1,2)=Y(G(I),G(I+1),G(I+2),G(I+3),H3H2)
      Z(I,1,3)=Y(G(I),G(I+1),G(I+2),G(I+3),H3H3)
      Z(I,2,1)=Y(G(I),G(I+1),G(I+2),G(I+3),2-H3H)
      Z(I,2,2)=Y(G(I),G(I+1),G(I+2),G(I+3),2-H3H2)
      Z(I,2,3)=Y(G(I),G(I+1),G(I+2),G(I+3),2-H3H3)
C
      I=3
      Z(I,1,1)=Y(G(I-1),G(I),G(I+1),G(I+2),1+H3H)
      Z(I,1,2)=Y(G(I-1),G(I),G(I+1),G(I+2),1+H3H2)
      Z(I,1,3)=Y(G(I-1),G(I),G(I+1),G(I+2),1+H3H3)
      Z(I,2,1)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H)
      Z(I,2,2)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H2)
      Z(I,2,3)=Y(G(I-1),G(I),G(I+1),G(I+2),3-H3H3)
C
      I=N-1
      Z(I,1,1)=Y(G(I-3),G(I-2),G(I-1),G(I),3+H3H)
      Z(I,1,2)=Y(G(I-3),G(I-2),G(I-1),G(I),3+H3H2)
      Z(I,1,3)=Y(G(I-3),G(I-2),G(I-1),G(I),3+H3H3)
      Z(I,2,1)=Y(G(I-3),G(I-2),G(I-1),G(I),5-H3H)
      Z(I,2,2)=Y(G(I-3),G(I-2),G(I-1),G(I),5-H3H2)
      Z(I,2,3)=Y(G(I-3),G(I-2),G(I-1),G(I),5-H3H3)
C
      I=N-2
      Z(I,1,1)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H)
      Z(I,1,2)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H2)
      Z(I,2,1)=Y(G(I-2),G(I-1),G(I),G(I+1),2+H3H3)
      Z(I,2,2)=Y(G(I-2),G(I-1),G(I),G(I+1),4-H3H)
      Z(I,2,3)=Y(G(I-2),G(I-1),G(I),G(I+1),4-H3H2)
      Z(I,2,3)=Y(G(I-2),G(I-1),G(I),G(I+1),4-H3H3)
C
```

```
      I=1
      Z(I,1,1)=Y(G(2),G(3),G(4),G(5),1-H3H)
      Z(I,1,2)=Y(G(2),G(3),G(4),G(5),1-H3H2)
      Z(I,1,3)=Y(G(2),G(3),G(4),G(5),1-H3H3)
      J=N-4
      Z(I,2,1)=Y(G(J),G(J+1),G(J+2),G(J+3),4+H3H)
      Z(I,2,2)=Y(G(J),G(J+1),G(J+2),G(J+3),4+H3H2)
      Z(I,2,3)=Y(G(J),G(J+1),G(J+2),G(J+3),4+H3H3)
C
      H2=(H-2*ERR)/2
      H2H=H2/H
      ERRH=ERR/H
      I=N
      Z(I,1,1)=-Y(G(2),G(3),G(4),G(5),1-ERRH)/ERR
      Z(I,1,2)=-Y(G(2),G(3),G(4),G(5),1-ERRH-H2H)/
     1          (ERR+H2)
      Z(I,1,3)=-Y(G(2),G(3),G(4),G(5),1-ERRH-2*H2H)/
     1          (ERR+2*H2)
      J=N-4
      Z(I,2,1)=Y(G(J),G(J+1),G(J+2),G(J+3),4+ERRH)/ERR
      Z(I,2,2)=Y(G(J),G(J+1),G(J+2),G(J+3),4+ERRH+H2H)/
     1          (ERR+H2)
      Z(I,2,3)=Y(G(J),G(J+1),G(J+2),G(J+3),4+ERRH+H2H*2)/


     1          (ERR+2*H2)
C
      DO 200 I=2,II
      Z(I,1,1)=-Z(I,1,1)/ERR2
      Z(I,1,2)=-Z(I,1,2)/ERR1
      Z(I,1,3)=-Z(I,1,3)/ERR
      Z(I,2,1)=Z(I,2,1) /ERR2
      Z(I,2,2)=Z(I,2,2) /ERR1
      Z(I,2,3)=Z(I,2,3) /ERR
  200 CONTINUE
C
      JJ=N-2
      DO 300 I=3,JJ
      HI2=(I-2)*H
      HN2=(II-I)*H
  300 ZZ(I)=QUAD(-G(I-1)/H,Z(I,1,1),Z(I,1,2),Z(I,1,3))+
     1       QUAD(G(I+1)/H,Z(I,2,1),Z(I,2,2),Z(I,2,3))+
     1       QUAD(-G(2)/HI2,-Z(1,1,1)/(HI2+H3),-Z(1,1,2)/
     1    (HI2+H32),-Z(1,1,3)/(HI2+H33))+
     1       QUAD(G(II)/HN2,Z(1,2,1)/(HN2+H3),Z(1,2,2)/
     1    (HN2+H32),Z(1,2,3)/(HN2+H33))
C
      I=2
      HN2=(II-I)*H
      ZZ(I)=SIMP(Z(N,1,1),Z(N,1,2),Z(N,1,3))+
     1       QUAD(G(I+1)/H,Z(I,2,1),Z(I,2,2),Z(I,2,3))+
     1       QUAD(G(II)/HN2,Z(1,2,1)/(HN2+H3),Z(1,2,2)/
     1    (HN2+H32),Z(1,2,3)/(HN2+H33))
C
      I=N-1
      HI2=(I-2)*H
      ZZ(I)=SIMP(Z(N,2,1),Z(N,2,2),Z(N,2,3))+
     1        QUAD(-G(I-1)/H,Z(I,1,1),Z(I,1,2),Z(I,1,3))t
     1       QUAD(-G(2)/HI2,-Z(1,1,1)/(HI2+H3),-Z(1,1,2)/
     1    (HI2+H32),-Z(1,1,3)/(HI2+H33))
C
      RETURN
      END
```

```
      SUBROUTINE SIMP(X,Z,N,II,H,H3,SUM)
      REAL*4 X(N), Z(II,2,3)
      QUAD(A,B,C,D)=A+3*B+3*C+D
      PI=3.141592654
      SUMA=0.
      SUMB=0.
      DO 10 I=3,II,2
      J=I+1
      SUMA=SUMA+X(I)
   10 SUMB=SUMB+X(J)
      SUM=0.3333333333/PI*H*(4*SUMA+2.*SUMB+X(2)-X(II))
      SUM=SUM+0.375/PI*H3*(QUAD(X(2),Z(1,1,1),Z(1,1,2),Z(
     11,1,3))+QUAD(X(II),Z(1,2,1),Z(1,2,2),Z(1,2,3)))
      RETURN
      END


      SUBROUTINE SIMPX(X,N,II,H,SUM)
      REAL*4 X(N)
      PI=3.141592654
      SUMA=0.
      SUMB=0.
      IF(II/2*2.EQ.II) GO TO 100
      DO 10 I=2,II,2
      J=I+1
      SUMA=SUMA+X(I)
   10 SUMB=SUMB+X(J)
      SUM=0.3333333333/PI*H*(4.*SUMA+2.*SUMB+X(1)-X(II))
      RETURN
C
  100 IF(II.EQ.2) GO TO 200
      SUM=0.375/PI*H*(X(1)+3*X(2)+3*X(3)+X(4))
      IF(II.EQ.4) RETURN
      DO 150 I=5,II,2
      J=I+1
      SUMA=SUMA+X(I)
  150 SUMB=SUMB+X(J)
      SUM=SUM+0.3333333333/PI*H*(4*SUMA+2*SUMB+X(4)-X(II))
      RETURN
  200 SUM=H*(X(1)+X(2))/PI*0.5
      RETURN
      END
```

```
      SUBROUTINE SIMPG(G ,N,NN,H,I,SUM,ZZ)
      REAL*4 G (N,N) ,ZZ(N)
      SIMP(A,B,C)=4.*A+B+C
      QUAD(A,B,C,D)=0.375*(A+3.*B+3.*C+D)
C
      SUM=0.
      IF(I/2*2.EQ.I) GO TO 100
      DO 10 J=3,NN,2
      IF(I.EQ.J) GO TO 10
      SUM=SUM+SIMP(G(I,J),G(I,J+1),G(I,J-1))
   10 CONTINUE
      SUM=0.3333333333333*H*SUM
      GO TO 700
C
  100 IA=I-1
      IB=I+1
      DO 150 J=3,NN,2
      IF(J.GE.IA-2.AND.J.LE.IB+2) GO TO 150
      SUM=SUM+SIMP(G(I,J),G(I,J+1),G(I,J-1))
  150 CONTINUE
      SUM=SUM*0.3333333333
C
      IF(IA.GT.3) GO TO 200
      IF(IA.LT.3) GO TO 300
      SUM=SUM+0.5*(G(I,IA)+G(I,IA-1))
      GO TO 300
  200 SUM=SUM+QUAD(G(I,IA),G(I,IA-1),G(I,IA-2),G(I,IA-3))
C
  300 IF(IB.LT.N-2) GO TO 400
      IF(IB.GT.N-2) GO TO 500
      SUM=SUM+0.5*(G(I,IB)+G(I,IB+1))
      GO TO 500
  400 SUM=SUM+QUAD(G(I,IB),G(I,IB+1),G(I,IB+2),G(I,IB+3))
C
  500 SUM=H*SUM
C
  700 SUM=SUM+ZZ(I)
      RETURN
      END
```

$ENTRY

APPENDIX E

```
C
C       SOLITARY WAVE
C       USING A CONFORMAL MAPPING AND RIEMANN-HILBERT
C            SOLUTION TO A MIXED-BOUNDARY-VALUE PROBLEM
C       WRITTEN BY T.H.LIM , FEB.18,1978.
C
        REAL*4 C(21),Y(21),SN(21),CS(21),T(21),TA(21),AG(21)
        REAL*4 GA(21,21),GB(21,21),G(21),ZZ(21),TB(21)
        REAL*4 Z(21,4),X(21),ST(21),STA(21)
        REAL*4 F2W(4),Y1W(4)
C
C
C       N=NUMBER OF POINTS
C       H=STEP SIZE
C       AG=ANGLE
C       T(I)=T VALUES ON THE INTERVAL (-1,0)
C       C=VELOCITY
C       G=LOG(Q)
C
C
C       SUBROUTINE CUBI---LAGRANGIAN CUBIC-INTERPOLATION
C       SUBROUTINE TRAPZ---TRAPEZODAL RULE
C       SUBROUTINE ODD---COMPUTE SINGULAR INTEGRAL
C       SUBROUTINE SIMP---COMPUTE X AND Y ALONG THE
C                         FREE SURFACE
C
        READ,(F2W(I),I=1,4)
        READ,(Y1W(I),I=1,4)
        DO 2000 I=1,4
 2000   PRINT,I,F2W(I),Y1W(I)
C
        N=21
        NN=N-1
        H=1./NN
        PI=3.141592654
        AG(1)=0.
        AG(N)=0.
        SN(1)=0.
        C(N)=1.
        G(N)=0.
        X(1)=0.
C
        ER=0.025
        ERR=ER*H
        H2=(H-ERR)/2.
        H2H=H2/H
        H2H2=H2H*2
        ERR1=ERR+H2
C
        DO 10 I=1,N
        T(I)=-1.+(I-1)*H
   10   ST(I)=SQRT(-T(I))
        DO 2001 K1=1,4
        F2=F2W(K1)
        RF=2./F2
        DO 2001 K2=1,4
        Y1=Y1W(K2)
        K12=(K1-1)*4+K2
```

```
      ISUM=0.
      DO 11 I=2,N
   11 ISUM=ISUM+(I-1)
      TEMP=Y1/ISUM
      Y(1)=Y1
      DO 12 I=2,N
   12 Y(I)=Y(I-1)-(I-1)*TEMP
      Y(N)=0.
      PRINT,' '
      PRINT,'Y(I)='
      DO 13 I=1,N
   13 PRINT810,I,Y(I)
C
      DO 15 I=1,NN
      TA(I)=1./T(I)
   15 STA(I)=SQRT(-TA(I))
      DO 18 I=2,NN
   18 TB(I-1)=(TA(I-1)-TA(I))*0.5
      TB(NN)=20.
      PRINT,'T,TA,TB='
      DO 16 I=1,N
   16 PRINT17,I,T(I),TA(I),TB(I)
   17 FORMAT(1X,I10,3F20.10)
C
C
      DO 999 MMM=1,10
      PRINT,' '
      PRINT,' '
      PRINT,'MMM=',MMM
C
      DO 20 I=1,NN
      Q(I)=SQRT(1.-RF*Y(I))
   20 G(I)=ALOG(Q(I))
C
      CALL CUBI(G,ZZ,N,H,H2,H2H,H2H2,ERR,ERR1,NN,Z)
C
      DO 30 I=2,NN
      DO 30 J=1,N
      IF(I.EQ.J) GO TO 30
      GA(I,J)=G(J)/(T(J)-T(I))/ST(J)
   30 CONTINUE
C
      DO 40 I=2,NN
      DO 40 J=1,NN
   40 GB(I,J)=G(J)/(TA(J)-T(I))/STA(J)
C
C
      DO 100 I=2,NN
      CALL ODD(GA,N,N,2,N-I,H,I,SUMA)
      CALL TRAPZ(GB,    TB,N,I,SUMB)
  100 AG(I)=(SUMA+SUMB+ZZ(I))/PI*ST(I)
C
      CS(1)=-1./T(1)/Q(1)
      DO 200 I=2,NN
      TEMP=T(I)*Q(I)
      CS(I)=-COS(AG(I))/TEMP
      SN(I)=-SIN(AG(I))/TEMP
      CALL SIMPX(SN,N,I,H,Y(I))
      CALL SIMPX(CS,N,I,H,X(I))
  200 Y(I)=Y(1)-Y(I)
```

```
C
      IF(MMM.GE.6) GO TO 1000
      PRINT,'I,Q(I),AG(I),X(I),Y(I)='
      DO 800 I=1,N
  800 PRINT820,I,Q(I),AG(I),X(I),Y(I)
  820 FORMAT(1X,I10,4F20.10)
  810 FORMAT(1X,I10,F20.10)
      GO TO 999
C
 1000 PRINT1001,MMM
 1001 FORMAT('1',I10)
      DO 1100 I=1,10
 1100 PRINT,' '
      PRINT1002,F2
      PRINT,' '
      PRINT1003,Y(1)
      PRINT,' '
      PRINT1004,Q(1)
 1002 FORMAT(13X,'F2=',F10.7)
 1003 FORMAT(13X,'A =',F10.7)
 1004 FORMAT(13X,'QB=',F10.7)
      PRINT,' '
      PRINT,' '
      PRINT1005
 1005 FORMAT(18X,'I',16X,'X(I)',18X,'Y(I)')
      PRINT,' '
      DO 1006 I=1,NN
      PRINT,' '
 1006 PRINT1007,I,X(I),Y(I)
 1007 FORMAT(1X,I18,2F22.7)
      DO 1008 I=1,4
 1008 PRINT,' '
      PRINT1009,K12
 1009 FORMAT(30X,'TABLE 7.',I2)
      PRINT1001,MMM
C
C
  999 CONTINUE

 2001 CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE CUBI(G,ZZ,N,H,H2,H2H,H2H2,ERR,ERR1,NN,Z)
      REAL*4 G(N),ZZ(N),Z(N,4)
      Y(A,B,C,D,X)=-(X-2)*(X-3)*(X-4)/6*A+
     1     0.5*(X-1)*(X-3)*(X-4)*B-
     1     0.5*(X-1)*(X-2)*(X-4)*C+(X-1)*(X-2)*(X-3)*D/6
      SIMP(A,B,C,H)=H*0.33333333*(A+4*B+C)
C
      II=NN-1
      DO 100 I=3,II
      Z(I,1)  =-Y(G(I-2),G(I-1),G(I),G(I+1),2+H2H)/ERR1
      Z(I,2)  =-Y(G(I-2),G(I-1),G(I),G(I+1),2+H2H2)/ERR
      Z(I,3)  =Y(G(I-1),G(I),G(I+1),G(I+2),3-H2H)/ERR1
  100 Z(I,4)  =Y(G(I-1),G(I),G(I+1),G(I+2),3-H2H2)/ERR
C
      I=2
      Z(I,1)  =-Y(G(I-1),G(I),G(I+1),G(I+2),1+H2H)/ERR1
      Z(I,2)  =-Y(G(I-1),G(I),G(I+1),G(I+2),1+H2H2)/ERR
      Z(I,3)  =Y(G(I-1),G(I),G(I+1),G(I+2),3-H2H)/ERR1
      Z(I,4)  =Y(G(I-1),G(I),G(I+1),G(I+2),3-H2H2)/ERR
C
      I=NN
      Z(I,1)  =-Y(G(I-2),G(I-1),G(I),G(I+1),2+H2H)/ERR1
      Z(I,2)  =-Y(G(I-2),G(I-1),G(I),G(I+1),2+H2H2)/ERR
      Z(I,3)  =Y(G(I-2),G(I-1),G(I),G(I+1),4-H2H)/ERR1
      Z(I,4)  =Y(G(I-2),G(I-1),G(I),G(I+1),4-H2H2)/ERR
C
      DO 200 I=2,NN
  200 ZZ(I)=SIMP(-G(I-1)/H,Z(I,1),Z(I,2),H2)+
     1     SIMP(G(I+1)/H,Z(I,3),Z(I,4),H2)
      RETURN
      END
```

```
      SUBROUTINE TRAPZ(GB,     TB,N,J,SUM)
      REAL*4 GB(N,N),     TB(N)
      SIMP(A,B,C,E)=0.33333333*E*(4.*A+B+C)
      SUM=0.
      NN=N-1
      NN1=NN-1
      DO 200 I=1,NN1
  200 SUM=SUM+TB(I)*(GB(J,I)+GB(J,I+1))
      RETURN
      END
```

```
      SUBROUTINE ODD (G,N,M,I1,I2,H,I,SUM)
      REAL*4 G(N,M)
      SIMP(A,B,C)=4.*A+B+C
      QUAD(A,B,C,D)=0.375*(A+3.*B+3.*C+D)
C
      SUM=0.
      IF(I/2*2.NE.I) GO TO 100
      DO 10 J=I1,I2,2
      IF(I.EQ.J) GO TO 10
      SUM=SUM+SIMP(G(I,J),G(I,J+1),G(I,J-1))
10    CONTINUE
      SUM=0.333333333333*H*SUM
      GO TO 700
C
100   IA=I-1
      IB=I+1
      DO 150 J=I1,I2,2
      IF(J.GE.IA-2.AND.J.LE.IB+2) GO TO 150
      SUM=SUM+SIMP(G(I,J),G(I,J+1),G(I,J-1))
150   CONTINUE
      SUM=SUM*0.3333333333
C
      IF(IA.GT.I1) GO TO 200
      IF(IA.LT.I1) GO TO 300
      SUM=SUM+0.5*(G(I,IA)+G(I,IA-1))
      GO TO 300
200   SUM=SUM+QUAD(G(I,IA),G(I,IA-1),G(I,IA-2),G(I,IA-3))
C
300   IF(IB.LT.I2 ) GO TO 400
      IF(IB.GT.I2 ) GO TO 500
      SUM=SUM+0.5*(G(I,IB)+G(I,IB+1))
      GO TO 500
400   SUM=SUM+QUAD(G(I,IB),G(I,IB+1),G(I,IB+2),G(I,IB+3))
C
500   SUM=SUM*H
C
700   RETURN
      END

      SUBROUTINE SIMPX(X,N,II,   H,SUM)
      REAL*4 X(N)
      PI=3.141592654
      SUMA=0.
      SUMB=0.
      IF(II/2*2.EQ.II) GO TO 100
      DO 10 I=2,II,2
      J=I+1
      SUMA=SUMA+X(I)
10    SUMB=SUMB+X(J)
      SUM=0.3333333333/PI*H*(4.*SUMA+2.*SUMB+X(1)-X(II))
      RETURN
C
100   IF(II.EQ.2) GO TO 200
      SUM=0.375/PI*H*(X(1)+3*X(2)+3*X(3)+X(4))
      IF(II.EQ.4) RETURN
      DO 150 I=5,II,2
      J=I+1
      SUMA=SUMA+X(I)
150   SUMB=SUMB+X(J)
      SUM=SUM+0.3333333333/PI*H*(4*SUMA+2*SUMB+X(4)-X(II))
      RETURN
200   SUM=H*(X(1)+X(2))/PI*0.5
      RETURN
      END

$ENTRY
```

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